Chapter 22

MSMYP2 General Topology

(22.1) Metric Spaces

(22.1.1) Preliminaries

Topology generalises much of real analysis into larger and more oddly behaved spaces. To this end it is important to understand basic results and concepts from set theory. Also, the familiar notation used for dealing with functions is changed slightly.

Functions

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, and $A \subset X$ and $b \subset Y$. The notation $f(x)$ has the usual meaning, but now define

$$f\colon \mathcal{P}(X) \to \mathcal{P}(Y) \quad \text{by} \quad f(A) = \{f(a) \in Y \mid a \in A\}$$

The inverse, $f^{-1}(B)$ can also be defined, but has a very different meaning to what is 'normal'.

$$f\colon \mathcal{P}(Y) \to \mathcal{P}(X) \quad \text{by} \quad f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

This defines the function $f^{-1}(B)$, the notation f^{-1} will not be used for anything else. The 'normal' inverse of f will be denoted by a different function such as h, say.

When acting on a set, f and f^{-1} have different domain and codomain than when acting on an number belonging to a set. Strictly speaking this makes f acting on a set a completely different function to f acting on a number. How, then, is f acting on sets converted to f acting on numbers? And vice versa?

Define the inclusion function $i_B\colon B \to Y$ by $y \mapsto y$. This seems quite odd, but is really quite straight forward. For example, it is required to apply the function g to an element $b \in B$. However, g has has its domain Y, so it is therefore necessary to convert b to an element of Y. Use the composite mapping $B \to Z$ given by $g \circ i_B$. In fact $f \circ i_A$ is called the restriction of f onto A.

Composition of functions works in the usual way, but care must be taken as to whether the functions composed have compatible codomain and domain—sets or numbers, and if sets does the inclusion function need using.

Functions acting on sets have a number of properties relating to preservation of properties such as inclusion, relative complement, intersection, and union. Although f does not preserve all these, f^{-1} does, and it is simple to show. As it is used in Theorem 12 the following lemma is proven.

Lemma 1 *The function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves inclusion in the sense that if $A \subset B \in \mathcal{P}(Y)$ then $f^{-1}(A) \subset f^{-1}(B)$.*

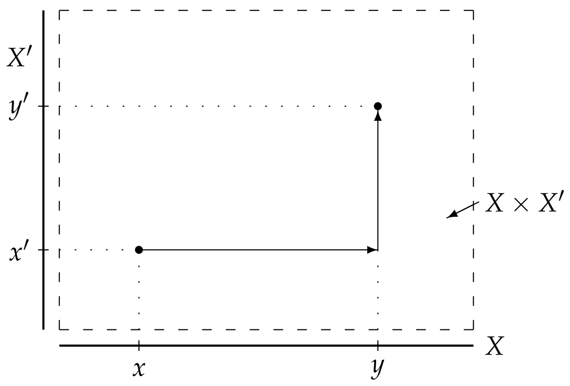

Figure 1: Graphical representation of the New York metric.

Proof. Suppose $A, B \in \mathcal{P}(Y)$ and $A \subset B$. Then

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\} \subset \{x \in X \mid f(x) \in B\} = f^{-1}(B)$$ □

(22.1.2) Metrics

Definition 2 *A metric is a function $d\colon X \times X \to \mathbb{R}$ ($X \neq \emptyset$) with the properties*

1. $d(x, x) = 0$
2. $d(x, y) > 0$ *when* $x \neq y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leqslant d(x, y) + d(y, z)$ *this is called the triangle inequality.*

A metric space is simply a space which has a metric defined on it, and thus is expressed as a pair (X, d). Worth mention are three particular metrics

- The discrete metric is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- The New York or taxi-cab metric on the set $X \times X'$ is defined by

$$d \times d'((x, x'), (y, y')) = d(x, y) + d(x', y')$$

  where (X, d) and $X', d')$ are metric spaces. This is illustrated in Figure 22.1.2, from which the nomenclature becomes apparent.

- Euclidean n-space, $\mathbb{E}^n$ is defined to be $\mathbb{R}^n$ with the Euclidean metric. The Euclidean metric is, as would be expected,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Subsets

Definition 3 *Let $B(a,r)$ denote the ball of radius r centred at a, so that*

$$B(a,r) = \{x \in X \mid d(x,a) < r\}$$

where (X,d) is a metric space.

Definition 4 *A set $O \subset X$ is open if for all $x \in O$ $\exists \varepsilon > 0$ such that $B(x,\varepsilon) \subset O$. A set P is closed if P^C is open.*

Certainly X is open since choosing $\varepsilon = 1$,

$$B(x,1) = \{y \in X \mid d(x,y) < 1\} \subset X$$

This may seem a bit odd, since with $\varepsilon = 1$ the ball could extend outside X. However, 'outside of X' does not exist. For precisely this reason, $(1,2]$ is an open subset of $[0,2]$ even though it is not an open subset of $\mathbb{R}$.

Through a quirk of logic*, $\emptyset$ is also open. Observe that

$$\forall x \in \emptyset \quad \exists \varepsilon > 0 \quad \text{such that} \quad B(x,\varepsilon) \subset \emptyset$$

Since there are no such x, this certainly always holds and hence $\emptyset$ is open.

However, $\emptyset$ open means that $X = \emptyset^C$ must be closed. Furthermore, the logic showing $\emptyset$ to be open also shows that it is closed. Both X and $\emptyset$ are both open and closed, and this property is not limited to these two special sets. It is also possible for sets to be neither open nor closed.

Theorem 5 *If O_i where $i \in \{1,2,\ldots,n\}$ are all open, then $\bigcap_{i\in I} O_i$ is open*

Proof. For any $x \in \bigcap_{i\in I} O_i$, $x \in O_i$ for all i.
since O_i is open, $\exists \varepsilon_i$ such that $B(x,\varepsilon_i) \subset O_i$.
Take $\varepsilon = \min_{i\in I}\{\varepsilon_i\}$ then $B(x,\varepsilon) \subset B(x,\varepsilon_i)$ for all i. Hence $B(x,\varepsilon) \subset \bigcap_{i\in I} O_i$. □

A similar result is true for the union of open sets. In this case I need not even be countable. If $x \in \bigcup_{i\in I} O_i$ then $\exists i$ such that $x \in O_i$, so

$$\exists \varepsilon > 0 \quad \text{such that} \quad B(x,\varepsilon) \subset O_i \subset \bigcup_{i\in I} O_i$$

Now using DeMorgan's laws it is evident that finite unions and arbitrary intersections of closed sets are closed.

Theorem 6 *For a metric space (X,d), any ball $B(x,r)$ is open.*

Proof. Let a be any point in $B(x,r)$, so $d(x,a) = s < r$ say. Consider now $B(a, r-s)$ and let y be any point in this ball, this is shown in Figure 22.1.2. Then $d(a,y) < r - s$.

$$\begin{aligned} d(x,y) &\leqslant d(x,a) + d(a,y) \quad \text{by the trianle inequality} \\ &\leqslant s + (r-s) = r \end{aligned}$$

So $y \in B(x,r)$, meaning that any point of $B(x,r)$ has an open ball arround it that is a subset of $B(x,r)$, i.e. $B(x,r)$ is open. □

*Suppose $A \Rightarrow B$ i.e. A is a sufficient condition for B—if B is observed then A must have happened. If A never happens (say A is "all camels have seven legs"), then B could happen or not happen, nothing can be deduced about the state of B.

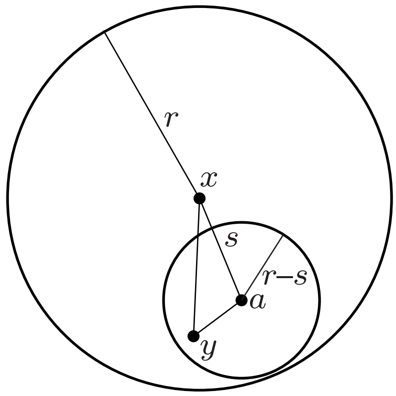

Figure 2: Diagram for proof of Theorem 6.

Limits

In analysis the Euclidean metric has been used without a second thought. Returning now to the definition of a limit and of convergence, it is possible to use any suitable metric.

Definition 7 *A 'limit' may commonly be used in one of three ways. The corresponding definitions are as follows.*

- *Let $\{a_n\}$ be a sequence in the metric space (X, d). $\{a_n\}$ has limit a as $n \to \infty$ if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that for } n > N \quad d(a_n, a) < \varepsilon$$

  Note that N may depend on ε.

- *Let $\{f_n\}$ be a sequence of functions, each $f_n \colon Y \to (X, d)$. $\{f_n\}$ is convergent to the function f if*

$$\forall \varepsilon > 0 \quad \forall y \in Y \quad \exists N \in \mathbb{N} \quad \text{such that for } n > N \quad d(f_n(y), f(y)) < \varepsilon$$

  Note that N may depend on both ε and y.

- *The sequence of functions $\{f_n\}$ is uniformly convergent to f if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that for } n > N \quad d(f_n(y), f(y)) < \varepsilon \quad \forall y \in Y$$

  Note that N may depend on ε.

The limits defined above work by saying "eventually, something happens". In the case of uniform convergence when N is found "something happens" at every value of y. This is not so with normal convergence, where N may need to be much larger for some values of y than for others.

The 'edge' of a set is often called its boundary—an obvious concept in $\mathbb{R}^2$ and $\mathbb{R}^3$. However, boundaries will be defined properly later.

Definition 8 *Let $A \subset X$.*

- *If every open ball around a point a (not necessarily in A) has in it at least one point $a' \neq a$ such that $a' \in A$, then a is a limit point of A.*
- *$a \in A$ is isolated if it is not a limit point of A.*
- *The closure of A, $\overline{A}$, is the smallest closed subset of X that contains A.*

Theorem 9 *The following are equivalent*

1. *A is a closed subset of X.*
2. $A = \overline{A}$
3. *All limit points of A are contained in A.*
4. *Every convergent sequence of elements of A has its limit in A.*

Proof. $1 \Rightarrow 2$ Suppose A is a closed subset of X and consider all the sets $F \subset X$ such that $A \subset F$. Since arbitrary unions of open sets are open, and the complement of an open set is a closed set, it follows that arbitrary intersections of closed sets are closed. Hence

$$A = \bigcap_{A \subset F \subset X} F$$

which is closed and must be the smallest closed subset of X that contains A, i.e. $A = \overline{A}$, as required. Furthermore the reverse implication also holds simply by reversing the argument.

$2 \Rightarrow 3$ Suppose $A = \overline{A}$ then A is closed and A^C is open. Take any $x \notin A$ so that $x \in A^C$ then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset A^C$. Therefore $B(x, \varepsilon) \cap A = \emptyset$, meaning that x is not a limit point of A. Hence A must contain all its limit points.

$3 \Rightarrow 4$ Suppose that all limit points of A belong to A, and let $\{a_n\}$ be a sequence of points in A that is convergent to a.
Now, either $a \in A$ or $a \notin A$. If $a \in A$ there is noting to show, so suppose $a \notin A$ and take any $B(a, \varepsilon)$. Since $\{a_n\} \to a$, $a_n \in B(a, \varepsilon)$ for $n > N$. Hence $a_{N+1} \in A$ and $a_{N+1} \in B(a, \varepsilon)$. But this defines a as a limit point, and A contains all its limit points. This contradicts the assumption $a \notin A$, so the proof is complete.

$4 \Rightarrow 1$ To prove this suppose that A is not closed and show that not every convergent sequence of elements of A has its limit in A.
Since A is not closed, A^C is not open and so there exists a point $x \in A^C$ for which no open ball $B(x, \varepsilon)$ is contained in A^C. Therefore every such ball must contain a point of A. Define the sequence

$$\begin{array}{lll} a_1 \in B(x, 1) & a_1 \in A & a_1 \neq x \in A^C \\ a_2 \in B(x, \frac{1}{2}) & a_2 \in A & a_2 \neq x \in A^C \\ \vdots & \vdots & \vdots \\ a_n \in B(x, \frac{1}{n}) & a_n \in A & a_n \neq x \in A^C \end{array}$$

then clearly $\{a_n\} \to x$ as $n \to \infty$, following readily from the definition of a limit. But $x \notin A$ and therefore the result is shown. □

Continuity

Clearly the definition of continuity of a function of $\mathbb{R} \to \mathbb{R}$ is readily modified for use in a metric space.

Definition 10 *Let (X, d) and X', d' be metric spaces and let $f : (X, d) \to (X', d')$ be a function. f is continuous at $a \in (X, d)$ if and only if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad d(x, a) < \delta \quad \Rightarrow \quad d'(f(x), f(a)) < \varepsilon$$

Furthermore, f is said to be uniformly continuous on (X, d) if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \quad f\left(B(x, \delta)\right) \subset B'(f(x), \varepsilon)$$

However, unlike functions of $\mathbb{R}$ to $\mathbb{R}$ it is possible to remove from the definition of continuity the requirement of a metric. A few results are required in order to produce the new definition.

Lemma 11 *$f(A) \subset B \Leftrightarrow A \subset f^{-1}(B)$, where $A \subset X$, the domain of f and $B \subset Y$, the codomain of f.*

Proof. ($\Rightarrow$) If $f(A) \subset B$ then $\forall a \in A$ $f(a) \in B$. $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ so certainly $A \subset f^{-1}(B)$.

($\Leftarrow$) $A \subset f^{-1}(B) \Rightarrow \forall a \in A$, $f(a) \in B$ so clearly $f(A) \subset B$. □

Note Lemma 11 is of the form "if one, then the other", and as such it certainly does not prove that $A = f^{-1}B$ or that $f(A) = B$. The situation is illustrated in Figure 22.1.2.

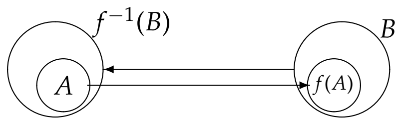

Figure 3: Diagram to illustrate Lemma 11.

Now, suppose that f is a continuous function. Then

$$\begin{aligned}
f \text{ continuous} &\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } d(x, a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon \\
&\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon) \\
&\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } f(B(a, \delta)) \subset B(f(a), \varepsilon) \\
&\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } B(a, \delta) \subset f^{-1}(B(f(a), \varepsilon)) \quad \text{from Lemma 11}
\end{aligned}$$

Now, since balls in metric spaces are open (Theorem 6) it seems reasonable to presume that the balls in the above equivalence could be replaced by any open sets.

Theorem 12 *$F^{-1}(O')$ is open in (X, d) for all open $O' \subset (X', d')$ if and only if $forall a \in X$ $\forall O'_{f(a)}$ $\exists O_a$ such that $O_a \subset f^{-1}(O'_{f(a)})$.*

Proof. ($\Rightarrow$) Take any $O'_{f(a)}$ then $f^{-1}(O'_{f(a)})$ is open (by hypothesis). Now, $a \in O'_{f(a)}$ so let $O_a = f^{-1}(O'_{f(a)})$ to give $O_a \subset f^{-1}(O'_{f(a)})$.

($\Leftarrow$) Take any open set O' in (X', d'). Let $a \in f^{-1}(O')$ then $f(a) \in O'$. By hypothesis $\exists O_a$ such that $O_a \subset f^{-1}(O')$. Take

$$O = \bigcup_{a \in f^{-1}(O')} O_a$$

then $O \subset f^{-1}(O')$ and is a union of open sets and thus is open. Therefore $O = f^{-1}(O')$ and must be open. □

What this theorem actually does is hide the use of the metric. Nevertheless, the working of the proof shows the metric is, in effect, still in use in the new equivalent definition of continuity. The merit of this theorem is that a consistent definition of continuity can be made where no metric exists but there is still sufficient structure to speak of open sets.

(22.2) Topological Spaces

(22.2.1) Topologies

Definition 13 *A topology $\mathcal{T}$ on a set X is a set of subsets of X such that*

- *$\emptyset$ and X are both in $\mathcal{T}$.*
- *If $A_1, A_2, \ldots, A_n \in \mathcal{T}$ then $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$.*
- *If $A_1, A_2, \cdots \in \mathcal{T}$ then $\bigcup_i A_i \in \mathcal{T}$.*

If $\mathcal{T}$ is a topology on X then $(X, \mathcal{T})$ is called a topological space.

In terms of a topological space, all the elements of $\mathcal{T}$ are defined to be open. As usual a set that isn't open is closed. Hence the notation O_a means $a \in O_a \in \mathcal{T}$.

Definition 14 *$\mathcal{B}$ is a base for a topology $\mathcal{T}$ if $\mathcal{B} \subset \mathcal{T}$ and every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$.*

For two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, it is quite possible that $\mathcal{T}_1 \subset \mathcal{T}_2$, say. In this case $\mathcal{T}_1$ is said to be weaker or coarser than $\mathcal{T}_2$.

Theorem 15 *An arbitrary intersection of topologies is a topology.*

Proof. To show this to be a topology is is simply necessary to verify the three properties.

- Since for all i, $\emptyset$ and X are in $\mathcal{T}_i$ it follows that $\emptyset$ and X are both in $\bigcap_i \mathcal{T}_i$.
- Say $\mathcal{T} = \bigcap_i \mathcal{T}_i$ and $O_1, O_2 \in \mathcal{T}$. Then $O_1, O_2 \in \mathcal{T}_i\ \forall i$.
  Hence $O_1 \cap O_2 \in \mathcal{T}_i\ \forall i$.
  Hence $O_1 \cap O_2 \in \mathcal{T}$.
- Let $O_j \in \mathcal{T}$ for all j in some set J.
  Then $O_j \in \rangle\ \forall i$.
  Hence $\bigcup_j O_j \in \mathcal{T}_i\ \forall i$.
  Hence $\bigcup_i O_j \in \mathcal{T}$. □

(22.2.2) Continuity & Limits

Theorem 12 provides a condition for continuity that does not directly rely on the use of a metric. In a topological space where no metric exists, this is simply taken as the definition of continuity.

Definition 16 *Let $f\colon (X, \mathcal{T}) \to (X', \mathcal{T}')$ be a function between topological spaces. f is continuous at $a \in X$ if*

$$\forall O'_{f(a)} \quad \exists O_a \quad \textit{such that} \quad O_a \subset f^{-1}\left(O'_{f(a)}\right)$$

If such a function is continuous everywhere in X then it is called a mapping. Furthermore, if f is one to one and onto (a bijection) then it is called a homeomorphism†.

As the sets of a topology $\mathcal{T}$ are, by definition, open. If a metric on the topological space $(X, \mathcal{T})$ gives the same open sets at $\mathcal{T}$ then $(X, \mathcal{T})$ is called metrisable.

Two topological spaces are called equivalent if they contain the same open spaces.

Definition 17 *The topological space $(X, \mathcal{T})$ is a Haussdorf space if for any two point $a, b \in X$ with $a \neq b$ there exist O_a and O_b in $\mathcal{T}$ such that $O_a \cap O_b = \emptyset$.*

In a Haussdorf space "there are enough open sets to separate points". This is important in defining limits, as if points cannot "be separated" a sequence may have more than one limit.

Lemma 18 *A metric space is a Haussdorf space*

Proof. Let (X, d) be a metric space and for $a, b \in X$ with $a \neq b$ say $d(a, b) = \varepsilon$. Define now $O_a = B(a, \frac{\varepsilon}{2})$ and $O_b = B(b, \frac{\varepsilon}{2})$. Then if $x \in O_a \cap O_b$ then

$$d(a, x) < \frac{\varepsilon}{2}$$
$$d(b, x) < \frac{\varepsilon}{2}$$
$$\text{hence by the triangle inequality, } d(a, b) < d(a, x) + d(x, b) < \varepsilon$$

which is a contradiction since $d(a, b) = \varepsilon$. Hence the result. □

A consequence of defining continuity in terms of the sets in a topology is that a function continuous between two topological spaces is not necessarily continuous between two other topological spaces. Observe that if $f \colon (X, \mathcal{T}) \to (X', \mathcal{T}')$ then if $\mathcal{T} = \mathcal{P}(X)$ it is certainly the case that $f^{-1}(O') \in \mathcal{T}$ for all $O' \in \mathcal{T}'$. Generalising the definition of continuity, limits can be defined.

Definition 19 *A sequence $\{a_n\}$ has limit a if $\forall O_a \; \exists N \in \mathbb{N} \; \forall n > N \; a_n \in O_a$*

Theorem 20 *The limit of a sequence in a Haussdorf topological space is unique.*

Proof. Let $(X, \mathcal{T})$ be a Haussdorf topological space and suppose the sequence $\{a_n\}$ has limits a and b where $a \neq b$.
Since $(X, \mathcal{T})$ is Haussdorf, $\exists O_a, O_b$ such that $O_a \cap O_b = \emptyset$.
Since $\{a_n\}$ has limit a, $a_n \in O_a$ for $n > N$.
Since $\{a_n\}$ has limit b, $a_n \in O_b$ for $n > N'$
Hence for $n > \max(N, N')$ $a_n \in O_a \cap O_b$. This contradicts $O_a \cap O_b = \emptyset$ and hence the result is shown. □

Consider a topology on $\mathbb{N}$ that consists of $\emptyset$ together with all the subsets of $\mathbb{N}$ that have finite complement. Using DeMorgan's laws it is easy to see this is not Haussdorf. Consider now the sequence defined by $a_n = n$. Consider now the open set about a point m, O_m. Since O_m has finite complement, $\exists N_m$ such that $a_n \in O_m$ for $n > N_m$. But this holds for all $m \in \mathbb{N}$, hence $\{a_n\}$ has limits of all values of $\mathbb{N}$.

Definition 21 *In the same way to a metric space, certain properties of sets can be defined.*

- *a is a limit point of the set A if all open sets containing a contain another point of A.*

†this should not be confused with a homomorphism which is a special kind of function that preserves binary operations. No such relationship is implied here.

- *The closure of A, $\overline{A}$ is the smallest closed set that contains A.*
- *The interior of A, A^o is the largest open set contained in A.*

From this it is evident that $\overline{A}$ is the intersection of all closed sets that contain A. Similarly A^o is the union of all open subsets of A. It is important to note that $\overline{A^c} \neq \overline{A}^c$. In fact $A^c \subset \overline{A^c}$ but $A^c \supset \overline{A}^c$.

The boundary of a set A can now be specified.

$$\text{Bdry}\, A = \overline{A} \cap \overline{A^c} \quad \text{so} \quad \overline{A} = A \cup \overline{A}$$

From this it can be shown that a set is closed if and only if it includes its boundary.

(22.2.3) Constructing Topologies

It is all very well defining topologies in some way, but new topologies can be constructed from existing ones, and certain processed may give rise to.

Subspace Topology

One way to define a new topology is to consider the inclusion function $i_A \colon (A, \mathcal{T}_A) \to (X, \mathcal{T})$ where $A \subset X$ and choose $\mathcal{T}_A$ to be the smallest topology such that this is continuous.

Note that at least one such topology exists, namely $\mathcal{T}_A = \mathcal{P}(A)$ which certainly makes i_A continuous. To make continuity, it is required that

$$\forall O \in \mathcal{T} \quad \exists O' \in \mathcal{T}' \quad \text{such that} \quad O' = i_A^{-1}(O)$$

Hence

$$i_A^{-1}(O) = \{x \in X \mid i_A(x) \in O\} \in \mathcal{T}$$

Now, if $x \notin A$ (remember $A \subset X$) then $i_A(x)$ is not defined. Hence

$$i_A^{-1}(O) = \{a \in A \mid i_A(a) \in O\} = O \cap A$$

Therefore define

$$\mathcal{T}_A = \{O \cap A \mid O \in \mathcal{T}\}$$

This is certainly the coarsest topology since the above shows it contains only the open sets required to make i_A continuous.

Product Topologies

Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces, and seek the coarsest topology on $X \times X'$ to make the projection functions continuous.

Recall the projection functions from $X \times X'$ are

$$p \colon X \times X' \to X \quad \text{defined by} \quad p \colon (x, x') \mapsto x$$
$$p' \colon X \times X' \to X' \quad \text{defined by} \quad p' \colon (x, x') \mapsto x'$$

The topology $\mathcal{P}(X \times X')$ certainly makes the projection functions continuous.

In order for the projection functions to be continuous, it is required that

$$p^{-1}(O) = O \times X' \in \mathcal{T} \times \mathcal{T}' \quad \forall O \in \mathcal{T}$$
$$(p')^{-1}(O') = X \times O' \in \mathcal{T} \times \mathcal{T}' \quad \forall O' \in \mathcal{T}'$$

Hence since any finite intersection of elements of a topology must be in the topology, taking the intersection gives $O \times O' \in \mathcal{T} \times \mathcal{T}'$. Take

$$\mathcal{B} = \{O \times O' \mid O \in \mathcal{T} \quad O' \in \mathcal{T}'\}$$

Lemma 22 *Let $\mathcal{B} \subset \mathcal{P}(X)$ then if*

1. *$X \in \mathcal{B}$ and $\emptyset \in \mathcal{B}$*
2. *If $B_2 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ then for all x in $B_2 \cap B_2$ there exists $B_x \in \mathcal{B}$ such that $B_x \subset B_1 \cap B_2$*

then $\mathcal{B}$ is a base for the smallest topology on X that contains $\mathcal{B}$.

Proof. First of all $\mathcal{B}$ is shown to be the base of a topology. That X and $\emptyset$ are in this topology is done by hypothesis. That the topology is closed under taking infinite unions is trivial since $\mathcal{B}$ is a base for the topology. For intersections, let A_1 and A_2 be in the topology, then

$$A_1 \cap A_2 = (\cup B_i) \cap (\cup B_j)$$
$$= \bigcup_{i,j} B_i \cap B_j$$
$$= \bigcup_{i,j} \bigcup_{x \in B_i \cap B_j} B_x$$

Secondly, it is shown that this is indeed the smallest topology. Let $\mathcal{B}$ be a base for $\mathcal{T}$ and let $\mathcal{T}'$ be another topology with $\mathcal{B} \subset \mathcal{T}'$.

$$T \in \mathcal{T} \Rightarrow T = \bigcup_i B_i \in \mathcal{T}'$$

Hence $\mathcal{T} \subset \mathcal{T}'$. □

Clearly $X \times X' \in \mathcal{B}, \emptyset \in \mathcal{B}$ and for any two sets in $\mathcal{B}$, $O_1 \times O_1'$ and $O_2 \times O_2'$ say

$$(x, x') \in (O_1 \times O_1') \cap (O_2 \times O_2') = (O_1 \cap O_2) \times (O_1' \cap O_2') = B_x$$

Hence the conditions of Lemma 22 are satisfied so $\mathcal{B}$ is a base for the smallest topology on $X \times X'$ that contains $\mathcal{B}$. By construction this precisely the coarsest topology to make the projection functions continuous.

Topological Union

In the case of the product topology the coarsest topology to make $(X, ?) \to (Y, \mathcal{T})$ continuous was sought. In the case of topological union, seek the finest topology to make a function of the form $(X, \mathcal{T}) \to (Y, ?)$ continuous. These are good ways to seek topologies as continuity is defined in terms of the sets $f^{-1}(O)$ where O is a subset of the codomain topology.

The topological union of the topological spaces $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ is the finest topology to make the inclusion functions continuous, i.e.

$$i_X : (X, \mathcal{T}) \to X \cup X' \quad \text{and} \quad i_{X'} : (X', \mathcal{T}') \to X \cup X'$$

Claim this is the topology

$$\{O \cup O' \mid O \in \mathcal{T} \quad O' \in \mathcal{T}'\}$$

Certainly $X \cup X'$ and $\emptyset = \emptyset_X \cup \emptyset_{X'}$ are in this set. Furthermore, for arbitrary unions,

$$\bigcup_i (O_i \cup O_i') = \left(\bigcup_i O_i\right) \cup \left(\bigcup_i O_i'\right)$$

which is a union of an element of $\mathcal{T}$ with an element of $\mathcal{T}'$ and hence is in the set. Now, for finite intersections,

$$(O_1 \cup O_1') \cap (O_2 \cup O_2') = (O_1 \cap O_2) \cup (O_1' \cap O_2')$$

which is a union of an element of $\mathcal{T}$ with an element of $\mathcal{T}'$ and hence is in the set. The properties for a topology are satisfied, hence this is a topology. What remains to be shown is that this is the finest topology to make the inclusion functions continuous.

Suppose there is a finer topology and $A \cup A'$ is in this topology but not the one already found. Then either $A \notin \mathcal{T}$ or $A' \notin \mathcal{T}'$ (or both) which makes the at least one of the inclusion functions discontinuous. Hence the result.

Identification Maps

Let $f\colon X \to Y$ be a function and $(X, \mathcal{T})$ be a topological space. Seek the finest topology on Y such that f is continuous. Claim this topology is the set $\mathcal{T}' = \{B \subset Y \mid f^{-1}(B) \in \mathcal{T}\}$. It is fairly obvious this is the correct choice, but this has to be shown.

Y and $\emptyset$ are in $\mathcal{T}'$ since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = (X)$. For finite intersections, say $B_1, B_2 \in \mathcal{T}'$ then

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

Now, for arbitrary unions, say $B_i \in \mathcal{T}'$ then

$$f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i)$$

hence $\mathcal{T}'$ is a topology. By construction $\mathcal{T}'$ is the finest topology making f continuous, since if C is an element of a finer topology, $f^{-1}(C) \notin \mathcal{T}$.

Combination Of Particular Topologies

Certain topological spaces are of special interest and have names identifying them. One such space is $\mathbb{I} = [0,1] \subset (\mathbb{R}, d)$. For example, $(-1, \frac{3}{4}) \cap [0,1] = [0, \frac{3}{4})$ is an open set in $\mathbb{I}$. Consider the mapping

$$\mathbb{I} \to \mathbb{C} \quad \text{defined by} \quad x \mapsto e^{2\pi i x}$$

Define $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$ then this mapping is onto. Furthermore, 0 in $[0,1]$ is identified with 1 in $[0,1]$ under this mapping. The product $\mathbb{I} \times \mathbb{S}$ then looks like the surface of a cylinder.

(22.2.4) Connectedness

Definition 23 *A topological space $(X, \mathcal{T})$ is connected if it is not the topological union of two disjoint sets.*

This definition essentially means that $\mathcal{T}$ cannot be split up into separate parts. If this could be done, it would be possible to draw a line through the set X, form topologies at either side of it, then take the topological union to create $\mathcal{T}$.

Theorem 24 *$(X, \mathcal{T})$ is not connected $\Leftrightarrow$ there exists a subset of X in $\mathcal{T}$ which is both open and closed.*

Proof.

$\Rightarrow$ If X is not connected then $X = A \dot{\cup} B$ where A and B are open (i.e. elements of $\mathcal{T}$) and non-empty.
It is clear that $A = B^c$ and so A (and B) are both open and closed in $\mathcal{T}$, as required.

$\Leftarrow$ Suppose $\emptyset \neq A \neq X$ and A is both open and closed in $\mathcal{T}$. Then clearly $X = A \dot{\cup} A^c$ so X is not connected. □

A topological space is connected if it its not the disjoint union of two of its open sets—this follows from above. But notice it is equally true that 'open' may be replaced with 'closed' in the preceding sentence. Indeed a number of conditions equivalent to connectedness may be established.

Theorem 25 *The topological space $(X, \mathcal{T})$ is connected if and only if*

1. *X and $\emptyset$ are the only subsets which are both open and closed.*
2. *Every continuous function from $(X, \mathcal{T})$ to the set $\{0, 1\}$ with the discrete topology (power set) is continuous.*

Proof.

1. Theorem 24 shows this to be so.
2. Equivalently, $(X, \mathcal{T})$ is disconnected if and only if there exists a map, as described, that is not constant. Suppose there exists a function $f \colon X \to \{0, 1\}$ which is not constant. Then $f^{-1}(0)$ and $f^{-1}(1)$ are open in $(X, \mathcal{T})$, so must be non-empty. Clearly $f^{-1}(0) \cap f^{-1}(1) = \emptyset$ and so, since f is a map, $X = f^{-1}(0) \dot{\cup} f^{-1}(1)$. Hence X is disconnected.
   Conversely, suppose X is disconnected, so $X = A \dot{\cup} B$ where A and B are non-empty. Define f by
   $$f \colon (X, \mathcal{T}) \to \{0, 1\} \qquad f \colon x \mapsto \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$
   then f is continuous and not constant. □

Even if a topology is connected, certain subspace topologies may not be. A subset A of X is said to be connected if it is connected in the subspace topology.

Lemma 26 *If $f \colon (X, \mathcal{T}) \to (X', \mathcal{T}')$ is a map and X is connected, then $f(X)$ is connected.*

Proof. It is logically equivalent to show that if $(X'\mathcal{T}')$ is disconnected then so is $(X, \mathcal{T})$. In this case $(X'\mathcal{T}') = A \dot{\cup} B$ and therefore $X = f^{-1}(A \dot{\cup} B)$. Since f^{-1} preserves union and intersection (intersection makes sure the two sets remain disjoint) $X = f^{-1}(A) \dot{\cup} f^{-1}(B)$ □

Theorem 27 *The product of connected spaces is connected.*

Proof. Suppose there exists a map from $X \times X'$ to $\{0, 1\}$ that is not constant. Then there exists points (x_0, x'_0) and (x_1, x'_1) such that $f(x_0, x'_0) = 0$ and $f(x_1, x'_1) = 1$.

Consider now the function

$$i_{x'_0} \colon X \to X \times X' \quad \text{defined by} \quad i_{x'_0} \colon x \mapsto (x, x'_0)$$

This function is clearly continuous since a set O gets mapped to $O \times \{x'_0\}$. It is also 1 to 1 and onto—it is a homeomorphism, so X becomes embedded in $X \times X'$.

Since f and $i_{x'_0}$ are continuous, so is $f \circ i_{x'_0}$ which maps X to $\{0,1\}$. Since X is connected and $f(x_0, x'_0) = 0$ it must be the case that $f \circ i_{x'_0}(x) = 0$ for all $x \in X$. (Otherwise a continuous function that is not constant has been found.)

Similarly $f \circ i_{x_1} : X' \to \{0,1\}$ is continuous, and since $f(x_1, x'_1) = 1$ by the same argument as above $f \circ i_{x_1}(x') = 1$ for all $x' \in X'$.

By the first case $f \circ i_{x'_0}(x_1) = f(x_1, x'_0) = 0$ but by the second case $f \circ i_{x_1}(x'_0) = f(x_1, x'_0) = 1$. Hence by contradiction the result is shown. □

Another example of a connected set is the the union of connected subsets A_1 and A_2 of X where $A_1 \cap A_2 \neq \emptyset$. Considering a map from this union to $\{0,1\}$, the map must be constant on each of A_1 and A_2. However, since $A_1 \cap A_2 \neq \emptyset$ it is readily deduced that any such map must be constant on $A_1 \cup A_2$.

Turning attention now to disconnected topological spaces, it is clear that a disconnected space may be a disjoint union of more than two other topological spaces.

Definition 28 *The component of x in the topological space $(X, \mathcal{T})$ where $x \in X$ is the union of every connected subset of X that contains x.*

The union described in this definition is not disjoint, as the intersection over all these sets is at least $\{x\}$. Components are therefore connected. Furthermore, the components of X form a partition of X since for $x, y \in X$ if

$$\operatorname{Comp} x \cap \operatorname{Comp} y \neq \emptyset \quad \text{then} \quad \exists z \in \operatorname{Comp} x \cap \operatorname{Comp} y$$

Consider $\operatorname{Comp} x \cup \operatorname{Comp} y$ then it is connected since it is the union of connected spaces which have non-empty intersection. Hence this is a connected set that contains x (and y) and so must be equal to both of $\operatorname{Comp} x$ and $\operatorname{Comp} y$.

Connectedness Of The Real Line

Having abstracted so far from reality it is of interest to see how results established thus far can be applied. First of all an apparently dimple question is answered: "what are the connected subsets of $\mathbb{R}$?"

Theorem 29 *Where $S \subset \mathbb{R}$ the following are equivalent*

1. *S is connected.*
2. *S is convex i.e. if $x, y \in S$ and $x < z < y$ then $Z \in S$.*
3. *S is an interval.*

Proof. Proof is given by establishing $1 \Leftrightarrow 2 \Leftrightarrow 3$, requiring 4 separate proofs.

$1 \Rightarrow 2$ It is equivalent to show $\neg 2 \Rightarrow \neg 1$.
Suppose S is not convex, then where $x, y \in S$ and $x < y$ then $\exists z \notin S$ with $x < z < y$. Hence

$$S = (-\infty, z) \cap S \cap (z, \infty) = ((-\infty, z) \cap S) \,\dot\cup\, (S \cap (z, \infty))$$

Now, $(-\infty, z)$ and (z, ∞) are open in $\mathbb{R}$ so S is a disjoint union of sets that are open in $\mathcal{T}_S$ and so is not connected.

$2 \Rightarrow 1$ Again, use the logical equivalent $\neg 1 \Rightarrow \neg 2$.
Suppose $S \subset \mathbb{R}$ is not connected, and hence there exists closed subsets F_1 and F_2 of $\mathbb{R}$ such that

$$S = (S \cap F_1) \dot{\cup} (S \cap F_2) \quad \text{where} \quad F_1 \cap F_2 \subset S^c \quad \text{and} \quad S \cap F_1 \neq \emptyset \neq S \cap F_2$$

Let $s_1 \in S \cap F_1$ and $s_2 \in S \cap F_2$ and without loss of generality, assume $s_1 < s_2$.
Consider $[s_1, s_2] \cap F_2$. s_1 is a lower bound and so there exists a greatest lower bound, s say. Now, $s > s_1$ since if $s = s_1$ then $s_1 \in F_2$ and bearing in mind $s_1 \in F_1$ and $s_1 \in S$ this would contradict $F_1 \cap F_2 \subset S^c$.
Consider $[s_1, s] \cap F_1$. Since s is an upper bound a least upper bound t, say, must exist. Similarly to above, $t < s$ since equality would contradict $F_1 \cap F_2 \subset S^c$.
Consider $[t, s]$.

- If $s = t$ then take $r = s = t$ which must be in $F_1 \cap F_2$ and so is not in S. Hence S is not convex.
- If $t < s$ take any r for which $s < r < t$ then $r \in [s_1, s_2]$ and so
  * $[s_1, s_2] \cap F_2$ has greatest lower bound s but $r < s$ so $r \notin F_2 \cap [s_1, s_2]$.
  * $[s_1, s] \cap F_1$ has least upper bound t but $r > t$ so $r \notin F_1 \cap [s_1, s_2]$.

  Hence $(t, s) \cap (F_1 \cup F_2) = \emptyset$. But $S \subset F_1 \cup F_2$ and so a point between s_1 and s_2 that is not in S has been found.

$2 \Rightarrow 3$ Let S be a convex subset of $\mathbb{R}$.

- If S is not bounded above or below then $S = (-\infty, \infty)$. For any $r \in \mathbb{R}$ there exists $a_1, a_2 \in \mathbb{R}$ which are not upper or lower bounds and such that $a_1 < r < a_2$ and hence by convexity $r \in S$.
- If S is not bounded below but is bounded above with least upper bound b then $S = (-\infty, b)$ or $S = (-\infty, b]$. If $r < b$ then there exists $a_1, a_2 \in \mathbb{R}$ such that $a_1 < r < a_2$ and hence by convexity S is an interval.
- If S is not bounded above but is bounded below with least upper bound a then $S = (a, \infty)$ or $S = [a, \infty)$. If $r > a$ then there exists $a_1, a_2 \in \mathbb{R}$ such that $a_1 < r < a_2$ and so by convexity S is an interval.
- If S is bounded below and above with supremum b and infimum a then $S = (a, b)$ or $S = [a, b]$. In either case S is an interval by convexity. □

It is quite surprising that so much work is required to prove something so 'obvious' as a convex set is connected. It is still possible to make a more intuitive description of the open subsets of $\mathbb{R}$.

Corollary 30 *If O is an open subset of $\mathbb{R}$ then O is a union of countably many open intervals.*

Proof. Consider the connected subsets of O—its components. Since they are connected in $\mathbb{R}$ they are intervals. In turn each component is a union of open intervals in $\mathbb{R}$ and hence each component is open and hence O is a disjoint union of open intervals. Now, every open interval contains a rational number, so O is a disjoint union of countable many intervals ($\mathbb{Q}$ is countable). □

(22.2.5) Path Connectedness

A path in a set X from a to b is simply a map α from $[0, 1]$ to X with $\alpha(0) = a$ and $\alpha(1) = b$.

Definition 31 *A topological space $(X, \mathcal{T})$ is path connected if for all $a, b \in X$ there exists a path in X from a to b. A subset of X is path connected if it is path connected in the subspace topology.*

Theorem 32 *A path connected topological space is connected.*

Proof. Instead the contrapositive statement is proven. Suppose, therefore, that $(X, \mathcal{T})$ is disconnected, so $X = A \dot\cup B$. Take $a \in A$ and $b \in B$ and suppose there exists a path α with $\alpha(0) = a$ and $\alpha(1) = b$. Then

- $\alpha^{-1}(A) \neq \emptyset$ and is an open subset of $[0, 1]$.
- $\alpha^{-1}(B) \neq \emptyset$ and is an open subset of $[0, 1]$.

Hence $[0, 1] = \alpha^{-1}(A) \dot\cup \alpha^{-1}(B)$ which contradicts the connectedness of $[0, 1]$. Therefore no such map can exist and so X is not path connected. □

Theorem 33 *Let $f\colon (X, \mathcal{T}) \to (X', \mathcal{T}')$ be a map. Then if X is path connected, so is the image if X.*

Proof. Take any two points in $f(X)$, say a' and b'. Then $a' = f(a)$ and $b' = f(b)$. Since X is path connected there exists a map $\alpha\colon [0, 1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Therefore $f \circ \alpha$ is a map from $[0, 1]$ to $f(X)$ with $f \circ \alpha(0) = a'$ and $f \circ \alpha(1) = b'$. □

Lemma 34 *Let $f\colon (X, \mathcal{T}) \to (X', \mathcal{T}')$ be a function between topological spaces. If $\mathcal{B}'$ is a base for $(X', \mathcal{T}')$ and if $f^{-1}(O') \in \mathcal{T}$ for all $O' \in \mathcal{B}'$ then f is a map.*

Theorem 35 *The product of two path connected topological spaces is path connected.*

Proof. Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be path connected topological spaces and let α and α' be paths in these spaces. Define the path β by

$$\beta\colon [0, 1] \to X \times X' \quad \text{by} \quad \beta\colon t \mapsto (\alpha(t), \alpha'(t))$$

Now let P be an open subset of $X \times X'$ and an element of the base. Then $\beta^{-1}(P) = \beta^{-1}(O \times O')$ for $O \in \mathcal{T}$ and $O' \in \mathcal{T}'$. Now,

$$\beta^{-1}(O \times O') = \{t \in [0, 1] \mid \alpha(t) \in O \text{ and } \alpha'(t') \in O'\} = \alpha^{-1}(O) \cap (\alpha')^{-1}(O')$$

which is an intersection of two open sets and so is open. Hence by Lemma 34 β is a map and the result is shown. □

It is fairly obvious that if α is a path connecting a to b, and β connects b to c then there exists a path connecting a to c. This path is defined by

$$\alpha\beta = \begin{cases} \alpha(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(2t - 1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Definition 36 *The path component of the point $x \in X$ is the set $y \in X$ such that there exists a path between x and y.*

Theorem 37 *The path components of X partition X.*

Proof. Let P and Q be path components and suppose $P \cap Q \neq \emptyset$, so $\exists x \in P \cap Q$. If $p \in P$ and $q \in Q$ then there exists a path from p to x and there exists a path from x to q. Hence $P = Q$. □

(22.2.6) Compactness

Definition 38 *A cover $\mathcal{C}$ for a set $A \subset X$ is a set of subsets of X such that $A \subset \bigcup_{C \in \mathcal{C}} C$.*

If all the elements of a cover are open sets, then the cover is referred to as an open cover. A subcover is, obviously, a subset of a cover that is itself a cover for the same set.

Definition 39 *A set A is a compact subset of X if for every open cover of A there exists a finite subcover of A.*

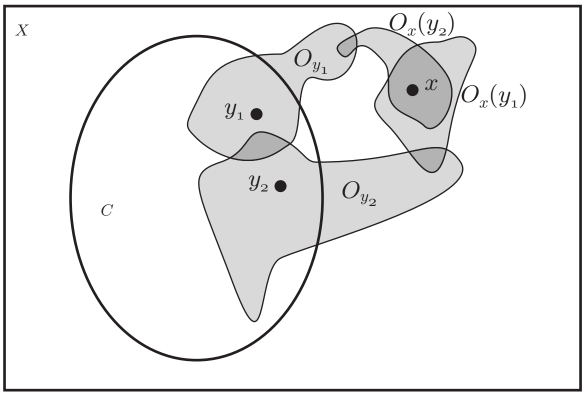

Figure 4: Diagram illustrating the proof of Theorem 42. x gets a different open set for each y, therefore take the intersection.

The form of this definition means that verifying it is a matter of taking some arbitrary open cover and showing it to have a finite subcover. This is the from of the next result.

Lemma 40 *If A is a finite subset of X then A is compact.*

Proof. Since A is finite, $A = \{a_1, a_2, \ldots, a_n\}$ say. Now take an open cover $\mathcal{C}$ and since it is a cover $\forall i\ \exists C_i \in \mathcal{C}$ such that $a_i \in C_i$. Hence $\{C_1, C_2, \ldots, C_n\}$ is a finite subcover for A. □

Theorem 41 *A closed subset of a compact space is compact.*

Proof. Let $(X, \mathcal{T})$ be a compact space and let $F \subset X$ be closed and consider any open cover $\mathcal{C}$ of F. Since F is closed F^c is open and so $\mathcal{C} \cup \{F^c\}$ is an open cover for X.
Since X is compact $\mathcal{C} \cup \{F^c\}$ has a finite subcover for X, $\{C_1, C_2, \ldots, C_n, F^c\}$ say. But $F \subset X$ and hence $\{C_1, C_2, \ldots, C_n\}$ is a finite cover for F. □

Theorem 42 *Compact subsets of Hausdorff spaces are closed.*

Proof. Let C be a compact subset of a Hausdorff space $(X, \mathcal{T})$. To show this is closed C^c must be shown to be open.
Take any $x \in C^c$ and any $y \in C$. Since X is Hausdorff open sets O_y and $O_x(y)$ can be chosen so that $O_y \cap O_x(y) = \emptyset$. Define

$$\mathcal{C} = \{O_y \mid y \in C\}$$

then $\mathcal{C}$ is an open cover for C. By compactness there exists a finite subcover $\{C_{y_1}, C_{y_2}, \ldots, C_{y_n}\}$. Now let

$$O_x = \bigcap O_x(y_i) \quad \text{so} \quad \forall i\ O_x \cap O_{y_i} = \emptyset$$

Therefore $\forall x \in C^c\ \exists O_x \subset C^c$ and hence C^c is open, meaning C is closed. □

It is worth noting that $\mathbb{R}$ is a Hausdorff space, and so all its compact sets are closed. In fact the compact sets are precisely the closed ones, as the next result (converse of Theorem 42) now shows.

Theorem 43 (Heine-Borel) *A closed interval of $\mathbb{R}$ is compact.*

Proof. Take any closed interval $[a, b] \subset \mathbb{R}$ and suppose there exists an open cover $\mathcal{C}$ which has no finite subcover. Divide $[a, b]$ into the two parts

$$\left[a, \frac{a+b}{2}\right] \quad \text{and} \quad \left[\frac{a+b}{2}, b\right]$$

then at least one of these does not have a finite subcover, say this is the set $[a_1, b_1]$. Repeating the process, $[a_1, b_1]$ has a subset $[a_2, b_2]$ which does not have a finite subcover. This process can continue indefinitely, yielding a sequence of intervals $[a_i, b_i]$ with

$$|a_i - b_i| = \frac{b-a}{2^i} \tag{44}$$

Also,

- $\{a_i\}$ is a non-decreasing sequence that is bounded above (by b) and so by the monotone convergence theorem it is convergent with limit a', say.
- $\{b_i\}$ is a non-increasing sequence that is bounded below, and so again by the monotone convergence theorem it must be convergent with limit b', say.

Now, by equation (44) $a' = b'$ and say this value is l. Since $l \in [a, b]$ $\exists O_l \in \mathcal{C}$, and by the topology on $\mathbb{R}$ there exists an open interval with l in it, $(l - \varepsilon_1, l + \varepsilon_2)$ say, which is a subset of O_l.
Choosing i to be sufficiently large, N say, then for $i > N$ $[a_i, b_i] \subset (l - \varepsilon_1, l + \varepsilon_2) \subset O_l$ but this contradicts the absence of a finite cover for $[a_i, b_i]$ and hence by contradiction the theorem is proven. □

Example 45 *In $\mathbb{R}$ the set $(0, 1)$ is not compact. To show this it is necessary to exhibit an open cover that has no finite subcover. Consider the cover*

$$\mathcal{C} = \left\{ \left(\frac{1}{n}, 1\right) \mid n \in \mathbb{N} \right\}$$

This is a cover since for any $r \in (0, 1)$ the Archimedean property ensures that $\exists n \in \mathbb{N}$ such that $n > \frac{1}{r}$. However, no finite subset of $\mathbb{C}$ can cover $(0, 1)$ since if

$$\mathcal{D} = \left\{ \left(\frac{1}{n_1}, 1\right), \left(\frac{1}{n_2}, 1\right), \dots \left(\frac{1}{n_k}, 1\right) \right\}$$

then $\bigcup \mathcal{D} = \left(\frac{1}{n_i}, 1\right)$ where $n_i = \max\{n_1, n_2, \dots n_k\}$ which certainly does not cover $(0, 1)$.

Similarly any interval of the form (a, b) is not compact. This is readily shown by considering the open cover

$$\mathcal{C} = \left\{ \left(a + \frac{1}{n}, b\right) \mid n \in \mathbb{N} \right\}$$

Finally for $\mathbb{R}$

Theorem 46 *Compact subsets of $\mathbb{R}$ are bounded.*

Proof. Let $A \subset \mathbb{R}$ that is not bounded above. Then

$$\mathcal{C} = \{(-\infty, n) \mid n \in \mathbb{N}\}$$

is an open cover that has no finite subcover and hence A is not compact. The other cases follow by symmetry. □

Hence for $\mathbb{R}$

- all compact subsets are closed (Theorem 42).
- all closed subsets are compact (Theorem 43).
- compact subsets are bounded (Theorem 46).

Though much has now been said about the compact subsets of $\mathbb{R}$ it remains to show some more general results. In particular that onto mappings preserve compactness, and results about subsets and product spaces.

Theorem 47 *If $(X, \mathcal{T})$ is a compact topological space and $f\colon (X, \mathcal{T}) \to (X', \mathcal{T}')$ is a mapping then $f(X)$ is compact.*

Proof. Let $\mathcal{C}'$ be any open cover of $f(X)$ then for any $C' \in \mathcal{C}'$, $f^{-1}(C')$ is open in X (since C' is open in $f(X)$). Certainly for any $x \in X$ there exists C' such that $x \in f^{-1}(C')$ and hence $\{f^{-1}(C') \mid C' \in \mathcal{C}\}$ is an open cover for X. But X is compact and so there exists a finite subcover $\{f^{-1}(C_1), f^{-1}(C_2), \ldots, f^{-1}(C_n)\}$ say. Therefore $\{C_1, C_2, \ldots, C_n\}$ is a finite subcover of $\mathcal{C}'$, hence the result. □

Theorem 48 *If $A \subset B \subset X$ then A is a compact subset of X if and only if A is a compact subset of $(B, \mathcal{T}_B)$.*

Proof. In this proof one must consider carefully what information is assumed and what must be proven, and how it relates to the definition of compactness.

- $\Rightarrow$ Suppose that A is a compact subset of X and consider any open cover of A in B. The cover is therefore of the form $\mathcal{C} = \{O_i \cap B \mid O_i \in \mathcal{T}\}$ and hence the set $\{O_i \mid O_i \cap B \in \mathcal{C}\}$ is an open cover of A as a subset of X. But A is a compact subset of X and hence there exists a finite subcover $\{O_1, O_2, \ldots, O_n\}$ say. Hence $\{O_1 \cap B, O_2 \cap B, \ldots, O_n \cap B\}$ is a finite subcover of $\mathcal{C}$.
- $\Leftarrow$ Suppose that A is a compact subset of B and that $\mathcal{C}$ is an open cover for A in X. Then where $O_i \in \mathcal{C}$ the set $\{O_i \cap B \mid O_i \in \mathcal{C}\}$ is a relatively open cover for A in B. But A is a compact subset of B and hence there exists a finite subcover $\{O_1 \cap B, O_2 \cap B, \ldots, O_n \cap B\}$ say. But therefore $\{O_1, O_2, \ldots, O_n\}$ is a finite subcover of $\mathcal{C}$. □

Theorem 49 *The product of two compact topological spaces is compact.*

Proof. Let $\mathcal{C}$ be an open cover for $X \times X'$, then for any $C \in \mathcal{C}$, C is a union of elements of the base of $X \times X'$, $\mathcal{B}$. Hence define

$$\mathcal{R} = \{O \times O' \in \mathcal{B} \mid \exists C \in \mathcal{C} \text{ with } O \times O' \subset C\}$$

which is itself a cover, and so for any point (x, x') there exists open sets containing them, $O'_{x'} \in \mathcal{T}'$ and $O_x(x') \in \mathcal{T}'$ with their product in $\mathcal{R}$.
Now, $\{O'_{x'} \mid x' \in X'\}$ is certainly an open cover for X' and so by compactness there exists a finite subcover $\{O'_{x'_i} \mid 1 \leqslant i \leqslant n\}$. Now let

$$O_x = \bigcap_{i=1}^{n} O_x(x'_i) \qquad \text{then} \qquad \bigcup_{i=1}^{n} \left(O_x \times O'_{x'_i}\right) = O_x \times X'$$

and each $O_x \times O'_{x'_i}$ is an element of $\mathcal{R}$ and hence $\exists C_i$ with $O_x \times O'_{x'_i} \subset C_i$. Therefore $O_x \times X'$ is covered by finitely many members of $\mathcal{C}$.

Consider $\{O_x \mid x \in X\}$ which is an open cover for X. By compactness there exists a finite subcover $\{O_{x_i} \mid 1 \leqslant i \leqslant m\}$ and so

$$X \times X' = \bigcup_{i=1}^{m} O_{x_i} \times X'$$

and hence $\mathcal{C}$ has a finite subcover. □