Chapter 19

MSMXA2 Vector Calculus

(19.1) Vector And Scalar Fields

(19.1.1) Index Notation

For the most part of this chapter, two and three dimensional problems will be under consideration. It is convenient to write, for example

$$\frac{\partial F_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

where the actual value of n is implicit by the context.

(19.1.2) Curves In Space

It is 'obvious' that a curve in 3 dimensions is a (continuous?) line that form some kind of a shape in $\mathbb{R}^3$.

Definition 1 *Let $[a, b] \subset \mathbb{R}$, then a path in $\mathbb{R}^n$ is a map $\mathbf{œ} \colon [a, b] \to \mathbb{R}^3$. $\mathbf{œ}(a)$ and $\mathbf{œ}(b)$ are called the endpoints of the path.*

Such a path is normally written in co-ordinate form as

$$\mathbf{œ} = (\sigma_1(t), \sigma_2(t), \sigma_3(t)) \quad \text{for} \quad t \in [a, b]$$

If the functions σ_i are differentiable, then the path $\mathbf{œ}$ is said to be differentiable, and the same for continuity.

Definition 2 *For a C^1 path $\mathbf{œ} \colon \mathbb{R} \to \mathbb{R}^3$ where $\mathbf{œ} = (x(t), y(t), z(t))$, the velocity is $\mathbf{v} = (x'(t), y'(t), z'(t))$, and the modulus of this is the speed, $S = \|\mathbf{œ}'(t)\|$.*

Note that the velocity vector is a tangent to the path, and its equation at the point $t = t_0$ is

$$\mathbf{l}(s) = \mathbf{œ}(t_0) + s\mathbf{œ}'(t_0)$$

provided $\mathbf{œ}'(t_0) \neq \mathbf{0}$.

Definition 3 *The length of a path $\mathbf{œ}$ is $l_{\mathbf{œ}} = \int_a^b \|\mathbf{œ}'(t)\| \, dt$.*

So far, a path has been parameterised by some kind of position in its domain. It can be far more convenient, however, to parameterise in terms of arc length.

Definition 4 *For a path*

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

the arc length s is defined by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{5}$$

The resulting equation

$$\mathbf{r}(s) = (x(s), y(s), z(s))$$

is called the intrinsic equation for the curve.

Definition 6 *The unit tangent vector* $\mathbf{T}$ *of a curve with intrinsic equation* $\mathbf{r}(s)$ *is the vector*

$$\mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$$

This is in fact always a unit vector. Consider tracing the curve, then the tangent vector is constantly changing direction, and hence the quantity $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is of interest. Now,

$$\begin{aligned} \mathbf{T} \cdot \mathbf{T} &= 1 \quad \text{since } \mathbf{T} \text{ is a unit vector} \\ 2\mathbf{T} \cdot \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} &= 0 \quad \text{by differentiating} \end{aligned}$$

so the derivative of the tangent vector is perpendicular to the tangent vector $\mathbf{T}$.

Definition 7 *Where* $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = k\mathbf{N}$, $\mathbf{N}$ *is the principal unit normal vector, and k is a non-negative function of s which is called the curvature of* **œ**.

Because $\mathbf{N}$ is not necessarily a unit vector, the scaling coefficient k is introduced. The quantity $\rho = \frac{1}{k}$ is the radius of curvature.

A tangent vector and a normal vector have now been found. However, since this is 3 dimensions, there must be another normal vector that is mutually perpendicular to $\mathbf{T}$ and $\mathbf{N}$. This is calculated using the cross product.

Definition 8 *The unit binormal vector is defined by the equation*

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

A relationship is now sought between $\mathbf{N}$ and $\mathbf{B}$.

Theorem 9 *The unit normal vector* $\mathbf{N}$ *and the unit binormal vector* $\mathbf{B}$ *are related by the equation*

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N}$$

where τ *is a function of s called the torsion of the curve.*

Proof. From the definition, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, and so differentiating,

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \times \mathbf{N} + \mathbf{T} \times \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s}$$

But $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = k\mathbf{N}$, so it is parallel to $\mathbf{N}$. Hence the first term is zero, giving

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = \mathbf{T} \times \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} \tag{10}$$

Furthermore since **B** is a unit vector,

$$\mathbf{B}\cdot\mathbf{B} = 1 \quad \text{so by differentiating,} \quad 2\mathbf{B}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = 0$$

Therefore $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ must lie in the plane spanned by **T** and **N**, the two perpendicular vectors i.e.

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = a\mathbf{T} + b\mathbf{N} \tag{11}$$

However, by (10), $\dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ is perpendicular to **T**, so it must be the case that $a = 0$. Hence putting $b = -\tau$,

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N}$$

□

The tangent **T** must always point 'along' the curve, but the two normal vectors may rotate round. The torsion, τ measures the rate (with respect to s) at which the binormal changes direction. For a planar curve it follows that $\tau = 0$. Notice that

- Since $\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is parallel to **N**, and both are unit vectors,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} &= k\mathbf{N}\\ \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\cdot\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} &= k\mathbf{N}\cdot\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\\ \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| &= k\end{aligned}$$

- Similarly, $\left\|\dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right\| = -\tau$.

Recall that for the three unit vectors **i**, **j**, and **k**,

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

Now **T**, **N**, and **B** are a right handed set, and since $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ it follows that

$$\begin{aligned}\mathbf{N} &= \mathbf{B}\times\mathbf{T} \quad \text{now differentiate,}\\ \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} &= \mathbf{B}\times\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} + \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\times\mathbf{T}\\ &= \mathbf{B}\times(k\mathbf{N}) + (-\tau\mathbf{N})\times\mathbf{T}\\ &= -k\mathbf{N}\times\mathbf{B} + \tau\mathbf{T}\times\mathbf{N}\\ &= -k\mathbf{T} + \tau\mathbf{B}\end{aligned}$$

The three equations

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = k\mathbf{N} \qquad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N} \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -k\mathbf{T} + \tau\mathbf{B}$$

are called the Serret-Frenet formulae.

Using the chain rule with the definition of arc length, equation (5), it is possible to produce all the above equations as functions of t rather than s.

(19.1.3) Fields

Definition 12 *A scalar field is a function* $f\colon \mathbb{R}^n \to \mathbb{R}$.

Definition 13 *A vector field is a function* $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$.

In few dimensions a scalar field is easy to visualise, an example is a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which defines a surface. A vector field is rather more difficult to imagine. Examples of a vector field might be the velocity of a fluid at different points in a container.

(19.1.4) Differentiation

Definition 14 *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a scalar field. The partial derivative of* f *with respect to* x_i *at the point* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *is*

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= \lim_{h\to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h} \\ &= \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}\end{aligned}$$

where $\mathbf{e}_i$ *is the ith standard basis vector.*

If a function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then it can be approximated locally by a line. Similarly, a scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point if it can be approximated locally by an $n-1$ dimensional hyperplane.

Definition 15 *The scalar field* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is differentiable at the point* $(x_{1_0}, x_{2_0}, \ldots, x_{n_0})$ *if all the partial derivatives with respect to the* x_i *exist, and*

$$\frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \left.\frac{\partial f}{\partial x_1}\right|_{\mathbf{x}_0}(x_1 - x_{1_0}) - \cdots - \left.\frac{\partial f}{\partial x_n}\right|_{\mathbf{x}_0}(x_n - x_{n_0})}{\|\mathbf{x} - \mathbf{x}_0\|} \to 0$$

as $x_i \to x_{i_0}$ *for all i.*

A vector field may by expressed in the form

$$\mathbf{F} = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$$

The derivative of such a function is a matrix,

$$T = \mathbf{D}\,\mathbf{F}(\mathbf{x}_0) = \left.\left(\frac{\partial f_i}{\partial x_j}\right)\right|_{\mathbf{x}_0}$$

Definition 16 *If* $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$ *is a vector field then* $\mathbf{F}$ *is differentiable at* $\mathbf{x}_0$ *if all the partial derivatives of each* f_i *exist, and*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0) - T(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

where

$$T = \mathbf{D}\,\mathbf{F}(\mathbf{x}_0) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}_0}$$

The derivative of the vector field is the matrix T. Having calculated the matrix T is is then necessary to check whether the vector field is indeed differentiable by evaluating the limit given above. There is, however, a convenient theorem that eliminates this requirements.

Theorem 17 *If $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$ is a vector field and if all the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist and are continuous in a neighbourhood of a point $\mathbf{x}$, then $\mathbf{F}$ is differentiable at $\mathbf{x}$.*

Properties Of Differentiation

Note that if a result holds for a vector field, then it must hold for a scalar field, which is just the special case of a vector field when the domain is $\mathbb{R}$.

Theorem 18 *Let $\mathbf{F}$ and $\mathbf{G}$ be vector functions that are differentiable at a point $\mathbf{x}_0$ and $c \in \mathbb{R}$. Then,*

1. $\mathbf{D}\,(c\mathbf{F}) = c\,\mathbf{D}\,\mathbf{F}$.
2. $\mathbf{D}\,(\mathbf{F}+\mathbf{G}) = \mathbf{D}\,\mathbf{F} + \mathbf{D}\,\mathbf{G}$. *The sum here is a sum of matrices.*
3. *For the the special case when $\mathbf{F}$ and $\mathbf{G}$ are scalar fields, $\mathbf{D}\,(FG) = F\,\mathbf{D}\,G + G\,\mathbf{D}\,F$ i.e. the product rule.*
4. *Again when $\mathbf{F}$ and $\mathbf{G}$ are scalar fields,*

$$\mathbf{D}\,\frac{F}{G} = \frac{G\,\mathbf{D}\,F - F\,\mathbf{D}\,G}{G^2}$$

Where $\mathbf{F}$ and $\mathbf{G}$ (or F and G) are evaluated at $\mathbf{x}_0$.

A version of the chain rule also holds for functions of many variables.

Theorem 19 (The Chain Rule) *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, and let $\mathbf{F}\colon V \to \mathbb{R}^p$ and $\mathbf{G}\colon U \to V$ be vector fields so that the composite function $\mathbf{F} \circ \mathbf{G}$ is defined. If $\mathbf{G}$ is differentiable at $\mathbf{x}_0$ and $\mathbf{F}$ is differentiable at $\mathbf{y}_0 = \mathbf{G}(\mathbf{x}_0)$, then*

$$\mathbf{D}\,\mathbf{F} \circ \mathbf{G}(\mathbf{x}_0) = \mathbf{D}\,\mathbf{F}(\mathbf{y}_0)\,\mathbf{D}\,\mathbf{G}(\mathbf{x}_0)$$

Higher order derivatives can be taken in the obvious way, and mixed partials are only equal if the function is continuously differentiable to the same order as the derivative.

Gradient

Since scalar fields give a singe number, whether this is increasing or decreasing — and how quickly it is doing so — is a meaningful question. This is not so with vector fields whether the result of the function is a vector.

Definition 20 *Let f be a differentiable scalar field. In three dimensions, the gradient of f at (x, y, z) is the vector*

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

It is easy to see how to extend this into more dimensions.

The derivative of a scalar field gives how quickly it changes in the direction of differentiation. This is limited to the the three axes, giving six directions with the careful introductions of ‘minuses’. There are of course a lot more than six directions, so it is of interest as to how to find the rate of change of the scalar field in any given direction — the directional derivative.

Theorem 21 *The directional derivative of the scalar field f at $\mathbf{x}_0$ in the direction of $\mathbf{v}$ is given by*

$$\nabla f(\mathbf{x}_0) \cdot \mathbf{v}$$

Proof. The derivative is being taken along the line $\mathbf{x}_0 + t\mathbf{v}$, so define the function $\mathbf{c}(t) = \mathbf{x}_0 + t\mathbf{v}$.

$$f(\mathbf{x}_0 + t\mathbf{v}) = f(\mathbf{c}(t))$$

now differentiating using the chain rule,

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \mathbf{D}\, f \cdot \mathbf{D}\,\mathbf{c} \\
&= \left(\frac{\partial f}{\partial c_1}, \frac{\partial f}{\partial c_2}, \frac{\partial f}{\partial c_3}\right) \cdot (v_1,\ v_2,\ v_3) \\
&= \frac{\partial f}{\partial c_1} v_1 + \frac{\partial f}{\partial c_2} v_2 + \frac{\partial f}{\partial c_3} v_3 \\
&= \nabla \cdot \mathbf{v}
\end{aligned}$$

Hence the result. □

Of course, the function $\mathbf{c}$ need not be a straight line — it is quite possible to find the rate of change of a scalar field along a path in its domain. Returning to the chain rule nd using it on the function $f(\mathbf{œ}(t))$,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \nabla f(\mathbf{œ}(t)) \cdot \mathbf{œ}'(t)$$

Theorem 22 *Provided $\nabla(\mathbf{x}) \neq 0$, the gradient vector ∇f points in the direction in which f is increasing fastest.*

Proof. Consider any unit vector $\mathbf{n}$, then $\mathbf{n} \cdot \nabla f = \|\nabla f\| \cos\theta$ where θ is the angle between the two vectors. Clearly the size of this dot product is maximal when $\theta = 0$ i.e. $\mathbf{n}$ and ∇f are parallel. Hence $\mathbf{n}$ points in the direction of maximal gradient only when it is parallel with ∇f. Hence the result. □

Now, along the level curves of a scalar field its value does not change at all. It follows therefore that the gradient vector evaluated at a point on a function of the form $f(x, y) = c$ must be normal to the curve. Furthermore, in three dimensions the gradient vector is normal to the level surface $f(x, y, z) = c$.

Curl & Divergence

For a vector field, the gradient is a meaningless quantity. Instead it is of interest as to whether the vectors get bigger or smaller — the divergence — and whether or not they twist like, say, a fluid might — the curl.

Definition 23 *Let $\mathbf{F}$ be a vector field. The curl of $\mathbf{F}$ is defined by*

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$

which is itself a vector field.

In Cartesian co-ordinates this has the obvious representation as a determinant,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

The cross product definition is of course valid an any co-ordinate system, but it is important to use the correct form of the vector differential operator, ∇.

As the name implies, the curl has interpretations to do with rotations and twisting.

If f is a scalar field then notice that $\nabla = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is a vector field, since it is a function of $\mathbb{R}^3$ into $\mathbb{R}^3$.

Theorem 24 *If $\mathbf{F} = \nabla f$ is a vector field, then $\nabla \times (\nabla f) = \mathbf{0}$ i.e. the curl of a gradient is zero.*

Proof. Writing $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ simply use the definition to give

$$\begin{aligned}\nabla \times (\nabla f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}\end{aligned}$$

But assuming that f is of class C^2 the order of differentiation in mixed partials doesn't matter, so clearly this gives the required result. □

As well as how the vector field curls, it is of interest as to whether it 'expands' or 'contracts', as shown in Figure 1. In the former case the vector field will map a region in its domain to a 'larger' region in its codomain. This property is the divergence.

Definition 25 *Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a vector field. The divergence of $\mathbf{F}$ is the quantity*

$$\mathbf{D}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar field.

A source, $\nabla \cdot \mathbf{F} > 0$ A sink, $\nabla \cdot \mathbf{F} < 0$

Figure 1: Interpretation of divergence

It was noted earlier that the curl of a vector field is itself a vector field.

Theorem 26 *For any C^2 vector field $\mathbf{F}$*, div curl $\mathbf{F} = 0$.

Proof. Use of the relationship $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is not allowed since ∇ is an operator. The result is easy to show by evaluating directly the equation $\nabla \cdot \nabla \times \mathbf{F}$ with $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. □

The property $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is readily shown by evaluating the two triple products. An alternative and rather unwieldy proof would involve multiplying out the given triple product involving the appropriate differential fractions. Note that

- a vector field whose divergence is always zero is called divergence free of incomprehensible.
- a vector field whose curl is always zero is called irrotational.

Notice that curl $\mathbf{F}$ is a vector field; this begs the question as to exactly what kind of vector fields can be expressed as the curl of some other vector field. As it happens, only incomprehensible vector fields can be expressed as the curl of some other vector field.

Notice that the divergence and curl are linear operators, this follows since the cross and dot product are linear operators.

Laplacian

Since the gradient of a scalar field is a vector field, what is the divergence of this vector field?

Definition 27 *Let f be a scalar field that is twice differentiable. The Laplacian of f is defined as*

$$\nabla^2 f = \nabla \cdot \nabla f = \operatorname{div} \operatorname{grad} f$$

In Cartesian co-ordinates this is simply the vector differential operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In some applications, the Laplacian of a vector function may be taken. This is the vector function with the Laplacian applied to each of the co-ordinates of the original vector function. It can be shown that

$$\nabla^2 \mathbf{F} = \nabla \left(\nabla \cdot \mathbf{F} \right) - \nabla \times \left(\nabla \times \mathbf{F} \right)$$

Proof of this is made by evaluating the right hand side.

Vector Differential Identities

Theorem 28 *For vector fields $\mathbf{F}$, $\mathbf{G}$, and scalar fields $\mathbf{H}$, and f and g,*

1. $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{G}(\mathbf{F} \cdot \nabla) + \mathbf{F}(\mathbf{G} \cdot \nabla) + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G})$
2. $\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f$
3. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$
4. $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$
5. $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}\nabla \cdot \mathbf{G} - \mathbf{G}\nabla \cdot \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$
6. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$
7. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$.
8. $\nabla \cdot (\nabla f \times \nabla g) = 0$.
9. $\mathbf{H} \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\mathbf{H} \times \mathbf{F}) + \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})$
10. $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = (\mathbf{F} \cdot \mathbf{H})\mathbf{G} - (\mathbf{F} \cdot \mathbf{G})\mathbf{H}$

These relations are readily shown from the appropriate definitions. It is at least worth verifying that the operators are being performed on the appropriate quantities.

(19.1.5) Orthogonal Co-Ordinate Systems

Cartesian co-ordinates are often the most simple to work in when deriving the assorted results discussed above. However, when it comes to describing any sort of circular motion, the Cartesian co-ordinate system is rather poor. Cylindrical and spherical co-ordinates address these problems but unfortunately the vector calculus results are not as easy.

Changing Co-Ordinate Systems

Suppose that it is desired to change from (x, y, z) co-ordinates to (u, v, w) co-ordinates where

$$x = f_1(u, v, w) \qquad y = f_2(u, v, w) \qquad z = f_3(u, v, w)$$

and suppose that these equations can be solved to produce

$$u = \overline{f}_1(x, y, z) \qquad v = \overline{f}_2(x, y, z) \qquad w = \overline{f}_3(x, y, z)$$

The position vector of a point is then

$$\mathbf{r} = f_1(u, v, w)\mathbf{i} + f_2(u, v, w)\mathbf{j} + f_3(u, v, w)\mathbf{k}$$

Setting say v and w constant and varying u produces parametric equations which define the u-line. If the u-line, the v-line, and the w-line are orthogonal, then the co-ordinate system (u, v, w) is called orthogonal. The unit vectors of this system are the tangents to these lines. Hence the following definition.

Definition 29 *Where the position vector of a point is given by*

$$\mathbf{r} = f_1(u, v, w)\mathbf{i} + f_2(u, v, w)\mathbf{j} + f_3(u, v, w)\mathbf{k}$$

define the three unit vectors of the (u, v, w) co-ordinate system by

$$\mathbf{e}_u = \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{e}_v = \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial v} \qquad \mathbf{e}_w = \frac{1}{h_3}\frac{\partial \mathbf{r}}{\partial w}$$

where

$$h_1 = \left\| \frac{\partial \mathbf{r}}{\partial u} \right\| \qquad h_2 = \left\| \frac{\partial \mathbf{r}}{\partial v} \right\| \qquad h_3 = \left\| \frac{\partial \mathbf{r}}{\partial w} \right\|$$

The three h functions are called the structure functions of the co-ordinate system (u, v, w) and their value may change from point to point.

Having found the unit vectors, it is now of interest as to how to express a point with co-ordinates known in (x, y, z) in terms of (u, v, w). Say $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ becomes $F_u\mathbf{e}_u + F_v\mathbf{e}_v + F_w\mathbf{e}_w$. Now, since the new unit vectors are orthogonal*

$$\begin{aligned} \mathbf{F} \cdot \mathbf{e}_u &= (F_u\mathbf{e}_u + F_v\mathbf{e}_v + F_w\mathbf{e}_w) \cdot \mathbf{e}_u \\ &= F_u \end{aligned}$$

*A co-ordinate system doesn't have to be orthogonal, but it is exceptionally difficult and so quite useless if it isn't but also using $\mathbf{e}_u = \frac{1}{h_1}\left(\frac{\partial f_1}{\partial u}\mathbf{i} + \frac{\partial f_2}{\partial u}\mathbf{j} + \frac{\partial f_3}{\partial u}\mathbf{k}\right)$,

$$\begin{aligned}\mathbf{F}\cdot\mathbf{e}_u &= F_1\mathbf{i}\cdot\mathbf{e}_u + F_2\mathbf{j}\cdot\mathbf{e}_u + F_3\mathbf{k}\cdot\mathbf{e}_u \\ &= \frac{1}{h_1}\left(F_1\frac{\partial f_1}{\partial u} + F_2\frac{\partial f_2}{\partial u} + F_3\frac{\partial f_3}{\partial u}\right)\end{aligned}$$

and similarly for the other two co-ordinates.

The vector calculus results can now be converted for use in alternative co-ordinate systems.

Theorem 30 *For a scalar field f expressed in an orthogonal co-ordinate system (u, v, w), the vector differential operator ∇ becomes*

$$\nabla = \mathbf{e}_u\frac{1}{h_1}\frac{\partial}{\partial u} + \mathbf{e}_v\frac{1}{h_2}\frac{\partial}{\partial v} + \mathbf{e}_w\frac{1}{h_3}\frac{\partial}{\partial w}$$

Proof. For a scalar field $f(x, y, z) = g(u, v, w)$, working from the definition of ∇,

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

which is to be expressed in the form

$$= G_u\mathbf{e}_u + G_v\mathbf{e}_v + G_w\mathbf{e}_w$$

where the Gs are to be determined. Now, from above

$$\begin{aligned}G_u &= \frac{1}{h_1}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\right) \\ &= \frac{1}{h_1}\frac{\partial f}{\partial u}\end{aligned}$$

and similarly for G_v and G_w. Hence

$$\nabla f = \mathbf{e}_u\frac{1}{h_1}\frac{\partial f}{\partial u} + \mathbf{e}_v\frac{1}{h_2}\frac{\partial f}{\partial v} + \mathbf{e}_w\frac{1}{h_3}\frac{\partial f}{\partial w}$$

As required. □

Notice that the unit vector is not differentiated, even though it will probably depend on the variables.

Applying this result it is easily deduced that

$$\mathbf{e}_u = h_1\nabla u \qquad \mathbf{e}_v = h_2\nabla v \qquad \mathbf{e}_w = h_3\nabla w$$

But u, v, and w are functions of x, y, and z; from earlier $u = \overline{f}(x, y, z)$. For constant u this defines a surface, and ∇u gives a normal vector to this surface. The unit vectors could therefore be defined as normals to these surfaces.

Having found how to change co-ordinate systems, it is now of interest as to what the usual vector differential operators become in alternative co-ordinate systems.

The Divergence

It would at first seem that finding the divergence is simply a case of applying the differential operator found in Theorem 30 to Definition 25. However, this is not the case, as this would not include in the differentiation process the basis vectors. This must be done since they are themselves functions of (u, v, w).

Beginning with Definition 25,

$$\nabla \cdot \mathbf{F} = \nabla \cdot (F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{w}) = \nabla \cdot F_u \mathbf{e}_u + \nabla \cdot F_v \mathbf{e}_v + \nabla \cdot F_w \mathbf{e}_w$$

Consider $\nabla \cdot F_u \mathbf{e}_u$—the other terms will follow by symmetry—then use $\mathbf{e}_u = h_1 \nabla u$.

$$\begin{aligned}
\nabla \cdot F_u \mathbf{e}_u &= \nabla \cdot (F_u h_1 \nabla u) \\
&= \nabla \cdot (F_u (h_2 \nabla v \times h_3 \nabla w)) \quad \text{since } \mathbf{e}_u = \mathbf{e}_v \times \mathbf{e}_w \\
&= \nabla \cdot \left(\underbrace{F_u h_2 h_3}\text{“}f\text{”} \underbrace{\nabla v \times \nabla w}\text{“}F\text{”} \right) \quad \text{now use identity 2 from page 8,} \\
&= (F_u h_2 h_3) \nabla \cdot (\nabla v \times \nabla w) + (\nabla v \times \nabla w) \cdot \nabla (F_u h_2 h_3)
\end{aligned}$$

but by identity 8 the first term is zero, hence

$$\begin{aligned}
&= (\nabla v \times \nabla w) \nabla \cdot (F_u h_2 h_3) \\
&= \frac{1}{h_2 h_3} \mathbf{e}_u \nabla \cdot (F_u h_2 h_3) \\
&= \frac{1}{h_2 h_3} \mathbf{e}_u \cdot \left(\mathbf{e}_u \frac{1}{h_1} \frac{\partial}{\partial u} (F_u h_2 h_3) + \mathbf{e}_v \frac{1}{h_2} \frac{\partial}{\partial v} (F_u h_2 h_3) + \mathbf{e}_w \frac{1}{h_3} \frac{\partial}{\partial w} (F_u h_2 h_3) \right) \\
&= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u} (F_u h_2 h_3)
\end{aligned}$$

and so by symmetry it is clear that the divergence is given by the formula

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u} (h_2 h_3 F_u) + \frac{\partial}{\partial v} (h_1 h_3 F_v) + \frac{\partial}{\partial w} (h_1 h_2 F_w) \right)$$

An alternative proof would involve starting with the Cartesian result $\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ and treat F_i as a composite function then use the chain rule. However, this is a rather lengthy method so it is preferable to memorise the vector differential identities.

The Curl

As with the divergence, the calculations are hampered by having to consider the basis vectors as functions. Nevertheless, direct contact with this can be ‘hidden’ using the vector differential identities.

Beginning with the definition of curl,

$$\nabla \times \mathbf{F} = \nabla \times (F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w) = \nabla \times (F_u \mathbf{e}_u) + \nabla \times (F_v \mathbf{e}_v) + \nabla \times (F_w \mathbf{e}_w)$$

considering only $\nabla \times (F_u \mathbf{e}_u)$ since the other tems will follow by symmetry,

$$
\begin{aligned}
\nabla \times (F_u \mathbf{e}_u) &= \nabla \times \Big(\underbrace{h_1 F_u}_{\text{“}f\text{”}} \underbrace{\nabla u}_{\text{“}\mathbf{F}\text{”}} \Big) \quad \text{now use identity 4} \\
&= \nabla (h_1 F_u) \times \nabla u + (h_1 F_u) \nabla \times (\nabla u) \\
&= \nabla (h_1 F_u) \times \nabla u \quad \text{since the curl of a gradient is the zero vector} \\
&= \left(\mathbf{e}_u \frac{1}{h_1} \frac{\partial}{\partial u} (h_1 F_u) + \mathbf{e}_v \frac{1}{h_2} \frac{\partial}{\partial v} (h_1 F_u) + \mathbf{e}_w \frac{1}{h_3} \frac{\partial}{\partial w} (h_1 F_u) \right) \times \mathbf{e}_u \frac{1}{h_1} \\
&= \begin{vmatrix} \mathbf{e}_u & \mathbf{e}_v & \mathbf{e}_w \\ \frac{1}{h_1} \frac{\partial}{\partial u} (h_1 F_u) & \frac{1}{h_2} \frac{\partial}{\partial v} (h_1 F_u) & \frac{1}{h_3} \frac{\partial}{\partial w} (h_1 F_u) \\ \frac{1}{h_1} & 0 & 0 \end{vmatrix} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_u & h_2 \mathbf{e}_v & h_3 \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 F_u & 0 & 0 \end{vmatrix}
\end{aligned}
$$

$$
\text{similarly} \quad \nabla \times (F_v \mathbf{e}_v) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_u & h_2 \mathbf{e}_v & h_3 \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ 0 & h_2 F_v & 0 \end{vmatrix}
$$

$$
\text{and} \quad \nabla \times (F_w \mathbf{e}_w) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_u & h_2 \mathbf{e}_v & h_3 \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ 0 & 0 & h_3 F_w \end{vmatrix}
$$

By summing these last three equations the result follows

$$
\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_u & h_2 \mathbf{e}_v & h_3 \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{vmatrix}
$$

The Laplacian

Finally, the fourth vector differential operator. The Laplacian is the divergence of the vector field formed by taking the del of a scalar field i.e. $\nabla^2 f = \nabla \cdot (\nabla f)$.

Working directly from the definition,

$$
\begin{aligned}
\nabla^2 f &= \nabla \cdot (\nabla f) \\
&= \nabla \cdot \left(\mathbf{e}_u \frac{1}{h_1} \frac{\partial f}{\partial u} + \mathbf{e}_v \frac{1}{h_2} \frac{\partial f}{\partial v} + \mathbf{e}_w \frac{1}{h_3} \frac{\partial f}{\partial w} \right)
\end{aligned}
$$

Now use the result for the divergence

$$
= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial w} \right) \right)
$$

This cannot be simplified further, so is the expression for the Laplacian. The fs may be removed giving an operator notation.

Cylindrical Polar Co-ordinates

Having produced a variety of results in general it important to see how they are applied to the two most important alternative orthogonal co-ordinate systems—cylindrical and spherical polar co-ordinates.

In Cylindrical polar co-ordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

so that

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k}$$

from which it is readily seen that

$$h_1 = \left\|\frac{\partial \mathbf{r}}{\partial r}\right\| = 1 \qquad h_2 = \left\|\frac{\partial \mathbf{r}}{\partial \theta}\right\| = r \qquad h_3 = \left\|\frac{\partial \mathbf{r}}{\partial z}\right\| = 1$$

and hence

$$\begin{aligned} \mathbf{e}_r &= \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial r} & \mathbf{e}_\theta &= \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial \theta} & \mathbf{e}_z &= \frac{1}{h_3}\frac{\partial \mathbf{r}}{\partial z} \\ &= \frac{\partial \mathbf{r}}{\partial r} & &= \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} & &= \frac{\partial \mathbf{r}}{\partial z} \\ &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} & &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} & &= z\mathbf{k} \end{aligned}$$

The results found above can be substituted into the various expressions for the differential operators and it is found that

$$\begin{aligned} \nabla &= \mathbf{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z} \\ \nabla\cdot\mathbf{F} &= \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial \theta}(F_\theta) + r\frac{\partial}{\partial z}(F_z)\right) \\ \nabla\times\mathbf{F} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} \\ \nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial u}\left(r\frac{\partial f}{\partial u}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial v^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

Spherical Polar Co-ordinates

The calculations run according in a similar way to those for cylindrical polar co-ordinates, except for requiring slightly more algebra—and more care. The results, though, are really quite straight forward and are set out below.

$$x = r\sin\phi\cos\theta \qquad y = r\sin\phi\sin\theta \qquad z = r\cos\phi$$

giving

$$h_1 = 1 \qquad h_2 = r\sin\phi \qquad h_3 = r$$

hence the unit vectors are

$$\begin{aligned} \mathbf{e}_r &= \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \\ \mathbf{e}_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \\ \mathbf{e}_\phi &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k} \end{aligned}$$

From the results collected above, it is simple to substitute into the expressions for the various differential operators and produce the following.

$$
\begin{aligned}
\nabla f &= \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r \sin\phi} \frac{\partial f}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r} \frac{\partial f}{\partial \phi} \\
\nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin\phi} \left(\frac{\partial}{\partial r} \left(r^2 \sin\phi F_r \right) + \frac{\partial}{\partial \theta} \left(r F_\theta \right) + \frac{\partial}{\partial \phi} \left(r \sin\phi \, F_\phi \right) \right) \\
\nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\phi} \begin{vmatrix} \mathbf{e}_r & r \sin\phi \mathbf{e}_\theta & r \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r \sin\phi F_\theta & r F_\phi \end{vmatrix} \\
\nabla^2 f &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial f}{\partial \phi} \right)
\end{aligned}
$$

(19.2) Integration

(19.2.1) Multiple Integrals

The multiple integral provides a way of integrating a scalar field over some subset of its domain. Their evaluation is discussed in Chapter **??**.

(19.2.2) Path & Line Integrals

Definition 31 *Let* $f\colon \mathbb{R}^3 \to \mathbb{R}$ *be a scalar field, and* $\boldsymbol{\oe}\colon [a,b] \to \mathbb{R}^3$ *where* $[a,b] \subset \mathbb{R}$ *be a path of class* C^1*, then*

$$\int_{\boldsymbol{\oe}} f \; \mathrm{d}s = \int_a^b f\left(\boldsymbol{\oe}(t)\right) \|\boldsymbol{\oe}'(t)\| \; \mathrm{d}t$$

where $\boldsymbol{\oe} = (x(t), y(t), z(t))$*. This is the path integral of* f *along* $\boldsymbol{\oe}$*.*

It is important to remember the path integral is evaluated with respect to arc length, and since $\frac{\mathrm{d}s}{\mathrm{d}t} = \left\| \frac{\mathrm{d}\boldsymbol{\oe}}{\mathrm{d}t} \right\|$ the factor $\|\boldsymbol{\oe}'\|$ is introduced to change the variables. Also, the expression $f(\boldsymbol{\oe})$ is meaningful since $\boldsymbol{\oe}\colon \mathbb{R} \to \mathbb{R}^3$.

The path integral has many practical uses, since if $\boldsymbol{\oe}$ is the shape if a wire, then integrating the scalar field $f = 1$ will find its length. Alternatively, f may be a density function so that the mass will be found.

The path integral can be defined in terms of a Riemann sum, taking the value of f on small segments of the path, i.e. Δs.

Path integrals are not applicable to vector fields, since the formulation of the integral is meaningless. Instead the line integral is defined as follows.

Definition 32 *Let* $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ *be a vector field, and* $\boldsymbol{\oe}\colon [a,b] \to \mathbb{R}^3$ *be a path of class* C^1*, then*

$$\int_{\boldsymbol{\oe}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_a^b \mathbf{F}\left(\boldsymbol{\oe}(t)\right) \cdot \boldsymbol{\oe}'(t) \; \mathrm{d}t$$

Again there are practical interpretations of the line integral, not least one of “force times distance” — something any GCSE student will claim to be “work”. If the direction of the force is always perpendicular to the direction of travel, then the integral must evaluate to zero. Resolving the force parallel and perpendicular to the line, it is evident that

$$\int_{\boldsymbol{œ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_a^b \left(\mathbf{F}\left(\boldsymbol{œ}(t)\right) \cdot \mathbf{T}(t)\right) \left\|\boldsymbol{œ}'(t)\right\| \ \mathrm{d}t$$

where $\mathbf{T}$ is the unit tangent vector along the path $\boldsymbol{œ}$. Note there is no perpendicular component since it is zero.

Consider a vector field as the force provided by the current of a river, and the path being the river. Traveling on the river in the direction of flow will be easy, since the force of the current will provide energy to move a boat. However, in the other direction the same energy is needed to counter the current and so stay still, and the same again to move at the same speed up the river.
It is plausible, therefore, that a line integral evaluated along a path $\boldsymbol{œ}\colon [a,b] \to \mathbb{R}^3$ should not be equal to, and in fact should have -1 times the value of the same line integral evaluated along $\boldsymbol{œ}\colon [b,a] \to \mathbb{R}^3$. This dependence on the direction of the path means the line integral is an oriented integral. If $\boldsymbol{œ}$ is a path described from a point a to a point b, and $\boldsymbol{æ}$ is the same path described from b to a, then

$$\int_{\boldsymbol{œ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = -\int_{\boldsymbol{æ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$$

The path $\boldsymbol{æ}$ is said to have the reverse orientation as $\boldsymbol{œ}$.

The idea of direction being reversed is difficult to express mathematically in terms of $\boldsymbol{œ}$ and $\boldsymbol{æ}$. However, it is readily seen that $\boldsymbol{œ}'(t) = -\boldsymbol{æ}'(t)$ i.e. the tangent vectors have opposite sense. Looking at the line integral formula and path integral formula — Definitions 32 and 31 respectively — it is clear why the line integral is oriented but the path integral is not.

The purpose of a path is to provide a set of points in space. The domain (in $\mathbb{R}$) of the path is not particularly important, and indeed the domain of the path can be changed, provided that it describes the same set of points in $\mathbb{R}^3$. Such a change is called a reparameterisation.

Definition 33 *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a C^1 bijection that maps an interval $[a,b]$ to an interval $[c,d]$. If $\boldsymbol{œ}\colon [a,b] \to \mathbb{R}^3$ is a piecewise C^1 path, then $\boldsymbol{æ} = \boldsymbol{œ} \circ h$ is a reparameterisation of $\boldsymbol{œ}$.*

A reparameterisation may preserve the direction of a path, or change it,

- If $h(a) = c$ and $h(b) = d$, then the reparameterisation is orientation preserving.
- If $h(a) = d$ and $h(b) = c$, then the reparameterisation is orientation reversing.

The possibility of the value of an integral changing sign is quite worrying, as the value of an integral is nolonger a well-defined quantity. Indeed, who is to say that it could not take even more values given some suitable reparameterisation. Although it seems quite trivial, the following theorem is, therefore, very important.

Theorem 34 *If $\boldsymbol{œ}$ and $\boldsymbol{æ}$ are parameterisations of the same curve, then for a vector field $\mathbf{F}$,*

$$\int_{\boldsymbol{œ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \pm \int_{\boldsymbol{æ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$$

where the direction is chosen according to whether the reparameterisation $\boldsymbol{æ}$ is orientation preserving or reversing respectively.

Proof. Working from the definition of the line integral,

$$\begin{aligned}
\int_{\text{æ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_a^b \mathbf{F}(\text{æ}) \cdot \text{æ}'(t)\ \mathrm{d}t \\
&= \int_a^b \mathbf{F}(\text{œ}(h(t))) \cdot \frac{\mathrm{d}\text{œ}(h(t))}{\mathrm{d}t}\ \mathrm{d}t \\
&= \int_a^b \mathbf{F}(\text{œ}(g(t))) \cdot \frac{\mathrm{d}\text{œ}(h(t))}{\mathrm{d}h(t)} \frac{\mathrm{d}h(t)}{\mathrm{d}t}\ \mathrm{d}t \quad \text{from the chain rule} \\
&= \int_{h(a)}^{h(b)} \mathbf{F}(\text{œ}(h(t))) \cdot \frac{\mathrm{d}\text{œ}(h)}{\mathrm{d}h}\ \mathrm{d}h \quad \text{by changing variables} \qquad (35)
\end{aligned}$$

The two cases of the reparameterisation being orientation preserving and orientation reversing are now considered in turn.

(a) If the reparameterisation is order preserving then $h(a) = c$ and $h(b) = d$. Hence equation (35) becomes

$$\int_{\text{æ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_c^d \mathbf{F}(\text{œ}(p)) \cdot \text{œ}'(p)\ \mathrm{d}p = \int_{\text{œ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$$

Hence the signs are the same.

(b) If the reparameterisation is order reversing then $h(a) = d$ and $h(b) = c$. Hence equation (35) becomes

$$\int_{\text{æ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_d^c \mathbf{F}(\text{œ}(p)) \cdot \text{œ}'(p)\ \mathrm{d}p = -\int_c^d \mathbf{F}(\text{œ}(p)) \cdot \text{œ}'(p)\ \mathrm{d}p = -\int_{\text{œ}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$$

Hence the signs are changed.

Since a reparameterisation can only be orientation preserving or orientation reversing the possible cases are exhausted and hence the theorem is proved. □

(19.2.3) Surface Integrals

The path and line integrals effectively reduce a three dimensional problem to one dimension, allowing integration over one variable. Surface integrals parameterise in two variables and so require double integration. In the same way that a path integral finds a length, the surface area of a scalar field finds area.

Definition 36 *A parameterised surface is a function* $\blacksquare\colon \mathbb{R}^2 \to \mathbb{R}^3$ *expressed in the form*

$$\blacksquare(u, v) = (x(u, v), y(u, x), z(u, v))$$

With the line and path integrals the use of a tangent line was required, however, a surface has a tangent plane. It is easy to see that tangent lines in the u and v directions at a point (u_0, v_0) are given by

$$\begin{aligned}
\mathbf{T}_u &= \left.\frac{\partial x}{\partial u}\right|_{(u_0,v_0)} \mathbf{i} + \left.\frac{\partial y}{\partial u}\right|_{(u_0,v_0)} \mathbf{j} + \left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)} \mathbf{k} \\
\mathbf{T}_v &= \left.\frac{\partial x}{\partial v}\right|_{(u_0,v_0)} \mathbf{i} + \left.\frac{\partial y}{\partial v}\right|_{(u_0,v_0)} \mathbf{j} + \left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)} \mathbf{k}
\end{aligned}$$

Since it is assumed that an orthogonal co-ordinate system is used, and since the surface is sufficiently smooth and 'well behaved' these two vectors span the tangent plane. A normal vector to the surface at the point (u_0, v_0) can therefore be found by taking the cross product $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$. Indeed, if this is nonzero then the surface is said to be smooth at (u_0, v_0).

Definition 37 *The integral of a scalar field f over a surface S which is parameterised by $\blacksquare\colon D \to \mathbb{R}^3$ where $D \subset \mathbb{R}^2$ is given by*

$$\int_S f(x,y,z)\ \mathrm{d}S = \int_D f(\blacksquare(u,v))\|\mathbf{T}_u \times \mathbf{T}_v\|\ \mathrm{d}u\ \mathrm{d}v$$

Note that if f is the scalar field $f(\mathbf{r}) = 1$ then the surface integral finds the area of the surface.

As line integrals are oriented, so are surface integrals of vector fields; the orientation of a plane is therefore of concern. Clearly a plane has two sides, one will be defined as the positive side or outside and the other as the negative side or inside. Notice that when dealing with closed shapes such as a sphere, the outside is the positive side. Again by analogy to the line integral,

- If $\dfrac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \mathbf{n}_1(P)$ where P is the point on the surface, then the parameterisation is orientation preserving.
- If $\dfrac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = -\mathbf{n}_1(P)$ where P is the point on the surface, then the parameterisation is orientation reversing.

However, not all surfaces are covered by these conditions: There are surfaces with only one side such as the Möbius strip, though this is usually of negligible concern.

There is a subtle difference between using the normal vector $\mathbf{n}$ and the unit normal vector $\mathbf{n}_1$. The objective of introducing the normal vector is to change "$\cdot \mathrm{d}\mathbf{S}$" to " $\mathrm{d}u\ \mathrm{d}v$". This gives

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n}_1\ \mathrm{d}S = \int_D \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}u\ \mathrm{d}v$$

Obviously the surface integral of a scalar field is independent of whether the parameterisation of the surface is orientation preserving ore reversing. Clearly this is not so with the surface integral of a vector field.

Theorem 38 *If S is a surface and $\blacksquare_1$ and $\blacksquare_2$ are two smooth parameterisations of S then*

(i) If $\blacksquare_1$ and $\blacksquare_2$ are orientation preserving then $\displaystyle\int_{\blacksquare_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\blacksquare_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

(ii) If $\blacksquare_1$ and $\blacksquare_2$ are orientation reversing then $\displaystyle\int_{\blacksquare_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -\int_{\blacksquare_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

(19.2.4) Integral Theorems

Domains, Surfaces & Volumes

In the previous section integrals were defined over volumes (the familiar triple integral), over surfaces (the surface integral which boils down to double integration), and along lines (which reduces to single integration). Clearly some of these integrals are easier to evaluate than others: The integral theorems of vector calculus provide ways to change between the various types of integral and can simplify immensely the process of integration. Note that 'V' is used for volumes, and 'S' for surfaces. 'D' represents a 'domain', usually in two dimensions corresponding to the surface integral can be performed as an iterated double integral.

The Divergence Theorem

Also known as Gausss Theorem, the Divergence Theorem links surface integrals with volume integrals. To be precise it links a volume integral to a surface integral over the bounding surface of the volume.

Theorem 39 (The Divergence Theorem) *Let S be a closed surface which bounds a volume V. If $\mathbf{F}$ is a vector field which is of class C^1 on both V and S then*

$$\int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V = \int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

where $\mathbf{n}$ is the outward pointing normal vector to the surface S.

Proof. Suppose the volume under consideration is bounded by two surfaces h_1 and h_2 which are functions $h \colon D \to \mathbb{R}$ where $D \subset \mathbb{R}^2$. The volume is therefore given by

$$V = \{(x,y,z) \mid (x,y) \in D,\ h_1(x,y) \leqslant z \leqslant h_2(x,y)\}$$

Hence evaluating the third component of the volume integral,

$$\begin{aligned}\int_V \frac{\partial F_3}{\partial z} \, \mathrm{d}z &= \int_{(x,y)\in D} \int_{h_1}^{h_2} \frac{\partial F_3}{\partial z} \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_D F_3\left(x,y,h_2(x,y)\right) - F_3\left(x,y,h_1(x,y)\right) \, \mathrm{d}x \, \mathrm{d}y \qquad (40)\end{aligned}$$

Consider now the surface integral. Where $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ the integral becomes

$$\int_S F_1 n_1 + F_2 n_2 + F_3 n_3 \, \mathrm{d}S$$

So that the surface is a proper function, it is thought of as two parts, the upper part and the lower part. This ensures that each point in the domain only maps to one point rather than two. Since the z component is under consideration at present, z is taken to be a function of x and y. However, when the other components are considered it will be necessary to move from the xy plane to the xz or yz plane. The task now is to find the normal vector. Consider the parameterisation of the upper part of the surface,

$$\begin{aligned}\mathbf{r}_U &= x\mathbf{i} + y\mathbf{j} + h_2(x,y)\mathbf{k} \\ \mathbf{r}_L &= x\mathbf{i} + y\mathbf{j} + h_1(x,y)\mathbf{k}\end{aligned}$$

By taking the cross product of the tangent vectors it is evident that

$$\mathbf{n}_U = \pm \frac{\frac{\partial \mathbf{r}_U}{\partial x} \times \frac{\partial \mathbf{r}_U}{\partial y}}{\left\| \frac{\partial \mathbf{r}_U}{\partial x} \times \frac{\partial \mathbf{r}_U}{\partial y} \right\|}$$

Now, $\dfrac{\partial \mathbf{r}_U}{\partial x} = \mathbf{i} + \dfrac{\partial h_2}{\partial x}\mathbf{k}$ and $\dfrac{\partial \mathbf{r}_U}{\partial y} = \mathbf{j} + \dfrac{\partial h_2}{\partial y}\mathbf{k}$. Hence

$$\begin{aligned}&= \pm \left(\frac{-h_{2x}}{\sqrt{1 + h_{2x}^2 + h_{2y}^2}}\mathbf{i} + \frac{-h_{2y}}{\sqrt{1 + h_{2x}^2 + h_{2y}^2}}\mathbf{j} + \frac{1}{\sqrt{1 + h_{2x}^2 + h_{2y}^2}}\mathbf{k} \right) \\ &= \frac{-h_{2x}}{\sqrt{1 + h_{2x}^2 + h_{2y}^2}}\mathbf{i} + \frac{-h_{2y}}{\sqrt{1 + h_{2x}^2 + h_{2y}^2}}\mathbf{j} + \frac{1}{\sqrt{1 + h_{2x}^2 + h_{2y}^2}}\mathbf{k}\end{aligned}$$

The last line follows because the normal vector is required to point outward, which in this case is 'up'. It is now necessary to change from integrating with respect to $\mathbf{S}$ to x and y. Now,

$$\begin{aligned} \cdot \mathrm{d}\mathbf{S}_U &= \mathbf{n}_U \, \mathrm{d}S_U \quad \text{By splitting } \mathbf{S} \text{ into a unit vector and scalar} \\ &= \mathbf{n}_U \left\| \frac{\partial \mathbf{r}_U}{\partial x} \times \frac{\partial \mathbf{r}_U}{\partial y} \right\| \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

Since the area of the surface element $\mathrm{d}S$ is the cross product of the tangent vectors.

$$\begin{aligned} &= \left(-h_{2x}\mathbf{i} + h_{2y}\mathbf{j} + \mathbf{k} \right) \\ \text{so} \quad \mathbf{k} \cdot \mathrm{d}\mathbf{S}_U &= \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

Similarly for the lower part of the surface,

$$\mathbf{n}_L = \frac{-h_{1x}}{\sqrt{1 + h_{1x}^2 + h_{2y}^2}}\mathbf{i} + \frac{-h_{1y}}{\sqrt{1 + h_{1x}^2 + h_{1y}^2}}\mathbf{j} + \frac{1}{\sqrt{1 + h_{1x}^2 + h_{1y}^2}}\mathbf{k}$$

from which it follows that $\mathbf{k} \cdot \mathrm{d}\mathbf{S}_L = - \, \mathrm{d}x \, \mathrm{d}y$. Using these results,

$$\begin{aligned} \int_S F_3\mathbf{k} \cdot \mathrm{d}\mathbf{S} &= \int_{S_U} F_3\mathbf{k} \cdot \mathrm{d}\mathbf{S}_U + \int_{S_L} F_3\mathbf{k} \cdot \mathrm{d}\mathbf{S}_L \\ &= \int_{S_U} F_3 \, \mathrm{d}x \, \mathrm{d}y - \int_{S_L} F_3 \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_D F_3 \left(x, y, h_2(x, y) \right) - F_3 \left(x, y, h_1(x, y) \right) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_V \frac{\partial F_3}{\partial z} \, \mathrm{d}V \end{aligned}$$

Comparing to (40), this has shown that the theorem holds for the z components, since manipulating both sides has produced the same expression. The process is now repeated for the other two components.

Where

$$V = \{(x, y, z) \mid (x, z) \in D_1, \; g_1(x, z) \leqslant y \leqslant g_2(x, z)\}$$

it follows through that

$$\int_V \frac{\partial F_2}{\partial y} \, \mathrm{d}V = \int_S F_2\mathbf{j} \cdot \mathrm{d}\mathbf{S}$$

And similarly where

$$V = \{(x, y, z) \mid (y, z) \in D_2, \; f_1(y, z) \leqslant y \leqslant f_2(y, z)\}$$

it follows through that

$$\int_V \frac{\partial F_1}{\partial x} \, \mathrm{d}V = \int_S F_1\mathbf{i} \cdot \mathrm{d}\mathbf{S}$$

Summing these components,

$$\begin{aligned} \int_V \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \, \mathrm{d}V &= \int_S (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot \mathrm{d}\mathbf{S} \\ \int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V &= \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \end{aligned}$$

Hence the result. □

Green's Theorems

Green's theorems are of rather less direct use than the divergence theorem. They both convert a surface integral to a volume integral.

Theorem 41 (Green's First Theorem) *Let f and g be scalar fields with $\nabla^2 f$ and $\nabla^2 g$ being defined throughout some volume V which is bounded by a simple closed surface S. If any discontinuities in $\nabla^2 f$ and $\nabla^2 g$ are finite and are confined to finitely many simple surfaces in V then*

$$\int_S f\nabla g \cdot \mathbf{n}\, \mathrm{d}S = \int_V f\nabla^2 g + \nabla f \cdot \nabla g\, \mathrm{d}V$$

This is sometimes written as

$$\int_S f\frac{\partial g}{\partial \mathbf{n}}\, \mathrm{d}S = \int_V f\nabla^2 g + \nabla f \cdot \nabla g\, \mathrm{d}V$$

where $\mathbf{n}$ is the outward unit normal vector of S.

Proof. From vector identity 2 on page 8, $\nabla(f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f$. Applying this to $\mathbf{H} = f\nabla g$,

$$\nabla \cdot \mathbf{H} = f\nabla \cdot (\nabla g) + \nabla f \cdot \nabla g = f\nabla^2 g + \nabla f \cdot \nabla g$$

Now applying the divergence theorem to $\mathbf{H}$,

$$\begin{aligned}
\int_S f\nabla g \cdot \mathbf{n}\, \mathrm{d}S &= \int_S \mathbf{H} \cdot \mathrm{d}\mathbf{S} \\
&= \int_V \nabla \cdot \mathbf{H}\, \mathrm{d}V \\
&= \int_V f\nabla^2 g + \nabla g \cdot \nabla g\, \mathrm{d}V \\
&= \int_V f\nabla^2 g + \nabla g \cdot \nabla g\, \mathrm{d}V
\end{aligned}$$

Hence the result. □

Theorem 42 (Green's Second Theorem) *Let f and g be scalar fields with $\nabla^2 f$ and $\nabla^2 g$ being defined throughout some volume V which is bounded by a simple closed surface S. If any discontinuities in $\nabla^2 f$ and $\nabla^2 g$ are finite and are confined to finitely many simple surfaces in V then*

$$\int_S f\mathbf{n} \cdot \nabla g - g\mathbf{n} \cdot \nabla f\, \mathrm{d}S = \int_S f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\, \mathrm{d}S = \int_V f\nabla^2 g - g\nabla^2 f\, \mathrm{d}V$$

where $\mathbf{n}$ is the outward unit normal vector to S.

Proof. From Green's first theorem,

$$\begin{aligned}
&\int_S f\frac{\partial g}{\partial \mathbf{n}}\, \mathrm{d}S = \int_V f\nabla^2 g + \nabla f \cdot \nabla g\, \mathrm{d}V \\
\text{and}\quad &\int_S g\frac{\partial f}{\partial \mathbf{n}}\, \mathrm{d}S = \int_V g\nabla^2 f + \nabla g \cdot \nabla f\, \mathrm{d}V
\end{aligned}$$

Subtracting these two results gives

$$\int_S f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\, \mathrm{d}S = \int_V f\nabla^2 g - g\nabla^2 f\, \mathrm{d}V$$

Which proves the theorem. □

The uses of Green's theorems are less obvious than those of the divergence theorem. However, one possible application is given in the example below.

Example 43 *Show that solutions to $\nabla^2 u = 0$ are unique on a domain D where $u = f$ on the boundary of D, ∂D.*

Proof. Solution Suppose that u and v are two solutions on D. Then $\nabla^2 u = \nabla^2 v = 0$. Since the Laplacian is a linear operator this gives $\nabla^2(u - v) = 0$.
By hypothesis $u = v = f$ on the surface S, the boundary of D. Hence $u - v$ has value 0 on S. Let $w = u - v$ and now apply Green's theorem.

$$\int_S w\nabla w \cdot \mathrm{d}\mathbf{S} = \int_V \nabla^2 w + \nabla w \cdot \nabla w \;\mathrm{d}V$$

But $w = 0$ on S and $\nabla^2 w = 0$ so

$$\begin{aligned} 0 &= \int_V \nabla w \cdot \nabla w \;\mathrm{d}V \\ &= \int_V \|\nabla w\|^2 \;\mathrm{d}V \end{aligned}$$

Now, $\|\nabla w\|^2 \geqslant 0$ so $\int_V \|\nabla w\|^2 \;\mathrm{d}V = 0 \iff \|\nabla w\|^2 = 0$ which means that w is a constant.
Hence $w = 0$ on the boundary of D and w is a constant. Hence $w = 0$ i.e. $u - v = 0$ so u is unique, as required. □

Stokes' Theorem

Stokes' theorem takes a similar form to the divergence theorem, but it connects line integrals to surface integrals. Lines and surfaces both being oriented quantities, it is important to define the correct orientation for the theorem.

Definition 44 *Let C be a closed curve, and S be a surface which is bounded by C. C and S are correspondingly oriented if S is on the left when C is traversed in an anticlockwise direction.*

Lemma 45 (Green's Theorem In The Plane) *Let $\mathbf{H} = H_1(x, y)\mathbf{i} + H_2(x, y)\mathbf{j}$ be a planar vector field having a continuous derivative in some region R. If the curve C is correspondingly oriented to, and bounds R, then*

$$\int_C \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \iint_R \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \;\mathrm{d}x \;\mathrm{d}y$$

Proof. Suppose that C is simple so that any line intersects it only twice (so it looks a bit like a circle). Suppose also that R is a region in the xy plane such that

$$R = \{(x, y) \mid a \leqslant x \leqslant b \; h_1(x) \leqslant y \leqslant h_2(x)\}$$

so that C can be expressed as

$$\begin{aligned} C_L &: \mathbf{r}_L = x\mathbf{i} + h_1(x)\mathbf{j} \qquad a \leqslant x \leqslant b \\ C_U &: \mathbf{r}_U = (b + a - x)\mathbf{i} + h_2(b + a - x)\mathbf{j} \qquad a \leqslant x \leqslant b \end{aligned}$$

Note that the curve is parameterised so that it is traversed anticlockwise, so C_L is first. The situation is shown diagrammatically in Figure 12.

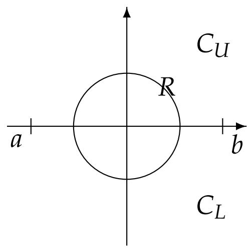

Figure 2: Simple representation of situation in Lemma 45.

Integrating over R,

$$\begin{aligned}\iint_R \frac{\partial H_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \int_{h_1}^{h_2} \frac{\partial H_1}{\partial y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_a^b H_1(x,h_2(x)) - H_1(x,h_1(x))\,\mathrm{d}x \qquad (46)\end{aligned}$$

Now,

$$\frac{\mathrm{d}\mathbf{r}_L}{\mathrm{d}x} = \mathbf{i} + h_1'\mathbf{j} \qquad \frac{\mathrm{d}\mathbf{r}_U}{\mathrm{d}x} = -\mathbf{i} - h_2'\mathbf{j}$$

This gives

$$\begin{aligned}\int_{C_U} H_1\mathbf{i}\cdot\mathrm{d}\mathbf{s} &= \int_a^b H_1(C_U)\cdot(-\mathbf{i} - h_2'\mathbf{j})\,\mathrm{d}x \\ &= -\int_a^b H_1(x,h_2(x))\,\mathrm{d}x\end{aligned} \qquad \begin{aligned}\int_{C_L} H_1\mathbf{i}\cdot\mathrm{d}\mathbf{s} &= \int_{C_L} H_1(C_L)\cdot(\mathbf{i} + h_1'\mathbf{j})\,\mathrm{d}x \\ &= \int_a^b H_1(x,h_1(x))\,\mathrm{d}x\end{aligned}$$

Using these in equation (46) gives

$$\int_a^b H_1(x,h_2(x)) - H_1(x,h_1(x))\,\mathrm{d}x = -\int_{C_U} H_1\mathbf{i}\cdot\mathrm{d}\mathbf{s} - \int_{C_L} H_1\mathbf{i}\cdot\mathrm{d}\mathbf{s} = -\int_C H_1\mathbf{i}\cdot\mathrm{d}\mathbf{s}$$

Hence

$$\iint_R \frac{\partial H_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = -\int_C H_1\mathbf{i}\cdot\mathrm{d}\mathbf{s} \qquad (47)$$

Now consider

$$R = \{(x,y) \mid c \leqslant y \leqslant d \; g_1(y) \leqslant x \leqslant g_2(y)\}$$

so that C can be expressed as

$$\begin{aligned}C_l &: \mathbf{r}_l = g_2(y)\mathbf{i} + y\mathbf{j} \qquad c \leqslant y \leqslant d \\ C_r &: \mathbf{r}_r = g_1(c+d-y)\mathbf{i} + (c+d-y)\mathbf{j} \qquad a \leqslant x \leqslant b\end{aligned}$$

In the same way as above it is found that

$$\iint_R \frac{\partial H_2}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_C H_2\mathbf{j}\cdot\mathrm{d}\mathbf{s} \qquad (48)$$

Subtracting equation (47) from (48) produces the result

$$\iint_R \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \int_C (H_1\mathbf{i} - H_2\mathbf{j})\cdot\mathrm{d}\mathbf{s} = \int_C \mathbf{H}\cdot\mathrm{d}\mathbf{s}$$

as required.

Now, in the case where part of C is parallel to one of the axes, it may be parameterised into a third part C_P, the part parallel to an axis. Now,

- If C_P is parallel to the y axis, then its parameterisation will have a constant $\mathbf{i}$ component. Hence the value of $\int_C H_1\mathbf{i}\cdot d\mathbf{s}$ does not change and the result still holds.
- If C_P is parallel to the x axis, then its parameterisation will have constant $\mathbf{j}$ component. Hence the value of $\int_C H_2\mathbf{j}\cdot d\mathbf{s}$ does not change and the result still holds.

Note that a horizontal part does not effect the calculation with $H_1\mathbf{i}$ because it is part of the h_1 function. The problem arises if there is a vertical part between where h_1 and h_2 should meet.

The results still hold, so the Lemma is proved. $\square$

Theorem 49 (Stokes' Theorem) *Let a vector field* $\mathbf{F}$ *and its curl be defined on a simple open surface* S*. If the correspondingly oriented curve* C *is the boundary of* S *then*

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$$

Proof. Suppose that $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ so

$$\nabla\times\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

Now suppose that S can be parameterised so that the position vector of a point as a function of (x, y) in some domain D is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

hence

$$\mathbf{T}_x = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k} \qquad \mathbf{T}_y = \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

Consider now the surface integral of some vector field $\mathbf{G}$,

$$\begin{aligned}\int_S \mathbf{G}\cdot d\mathbf{S} &= \iint_{(x,y)\in D} \mathbf{G}\cdot\|\mathbf{T}_x\times\mathbf{T}_y\|\ dx\ dy \\ &= \iint_D -\frac{\partial f}{\partial x}G_1 - \frac{\partial f}{\partial y}G_2 + G_3\ dx\ dy\end{aligned}$$

Now put $\mathbf{G} = \nabla\times\mathbf{F}$ to get

$$\int_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_D \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial f}{\partial x}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial f}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\ dx\ dy \tag{50}$$

Now consider the line integral of $\mathbf{F}$ round the boundary of S. The boundary of S can be thought of as the image under f of the boundary of D. If the boundary of D can be parameterised as

$$\boldsymbol{œ}\colon \mathbb{R}\to\mathbb{R}^2 \qquad \boldsymbol{œ}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

then it is evident that C is given by

$$\boldsymbol{æ}\colon \mathbb{R}\to\mathbb{R}^3 \qquad \boldsymbol{æ}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + f\left(x(t)\mathbf{i} + y(t)\mathbf{j}\right)\mathbf{k}$$

Hence from the definition of a line integral

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{æ}(t))\cdot \mathbf{æ}'(t)\,\mathrm{d}t \\
&= (F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot\left(\frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i}+\frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j}+\left(\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right)\mathbf{k}\right)\,\mathrm{d}t \\
&= \int_a^b \left(F_1+F_3\frac{\partial f}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}+\left(F_2+F_3\frac{\partial f}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_{\mathbf{œ}} \mathbf{H}\cdot\mathrm{d}\mathbf{s}
\end{aligned}$$

The last line follows from the definition of a line integral in two dimensions. It is taken round the boundary of D and

$$\mathbf{H}=\left(F_1+F_3\frac{\partial f}{\partial x}\right)\mathbf{i}+\left(F_2+F_3\frac{\partial f}{\partial y}\right)\mathbf{j}$$

Green's theorem in the plane is now used. Calculating the required partial derivatives, and remember that each F_i is a function of x, y, and $z=f(x,y)$,

$$\begin{aligned}
\frac{\partial H_2}{\partial x} &= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x}+F_3\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial x}\left(\frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial z}\frac{\partial f}{\partial x}\right) \\
\frac{\partial H_1}{\partial y} &= \frac{\partial F_1}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}+F_3\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial f}{\partial y}\left(\frac{\partial F_3}{\partial y}+\frac{\partial F_3}{\partial z}\frac{\partial f}{\partial y}\right)
\end{aligned}$$

Now returning to Green's theorem in the plane, and the calculations preceding it,

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{s} &= \int_{\mathbf{œ}}\mathbf{H}\cdot\mathrm{d}\mathbf{s} \\
&= \iint_D \frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_D \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x}+F_3\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial x}\left(\frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial z}\frac{\partial f}{\partial x}\right)-\frac{\partial F_1}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}-F_3\frac{\partial^2 f}{\partial x\partial y} \\
&\qquad -\frac{\partial f}{\partial y}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_3}{\partial z}\frac{\partial f}{\partial y}\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_D \frac{\partial z}{\partial x}\left(\frac{\partial F_2}{\partial z}-\frac{\partial F_3}{\partial y}\right)+\frac{\partial z}{\partial y}\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

This is precisely the same as equation (50) and hence the theorem is proved. □

(19.3) Conservative Fields

(19.3.1) Properties Of Conservative Fields

Definition 51 *A vector field* $\mathbf{F}$ *is conservative if there exists a scalar field* U *such that* $\mathbf{F}=\nabla U$*.* U *is called the potential.*

This special class of vector functions appear in many physical applications, some of which are discussed later. Given this simple condition a surprising number of results can be deduced.

Theorem 52 *Let* U *be a scalar field,* $U\colon \mathbb{R}^3\to\mathbb{R}$*, and let* $\mathbf{F}=\nabla U$*. If* $\mathbf{œ}\colon\mathbb{R}\to\mathbb{R}^3$ *is a* C^1 *path, then*

$$\int_{\mathbf{œ}}\mathbf{F}\cdot\mathrm{d}\mathbf{s}=U\left(\mathbf{œ}(b)\right)-U\left(\mathbf{œ}(a)\right)$$

Proof. From the definition of a line integral and of $\mathbf{F}$,

$$\int_{\boldsymbol{\oe}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_a^b \nabla U(\boldsymbol{\oe}(t)) \cdot \frac{\mathrm{d}\boldsymbol{\oe}}{\mathrm{d}t}\, \mathrm{d}t$$

But notice that $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(U(\boldsymbol{\oe}(t))\right) = \nabla U(\boldsymbol{\oe}(t)) \cdot \dfrac{\mathrm{d}\boldsymbol{\oe}}{\mathrm{d}t}$ and so putting $\frac{\mathrm{d}U_1}{\mathrm{d}t} = \nabla U(\boldsymbol{\oe}(t)) \cdot \frac{\mathrm{d}\boldsymbol{\oe}}{\mathrm{d}t}$

$$\begin{aligned}
&= \int_a^b \frac{\mathrm{d}U_1}{\mathrm{d}t}\, \mathrm{d}t \\
&= U_1(b) - U_1(a) \\
&= U\left(\boldsymbol{\oe}(b)\right) - U\left(\boldsymbol{\oe}(a)\right)
\end{aligned}$$

□

Theorem 53 *Let $\mathbf{F}$ be a vector field of class C^1, except possibly at finitely many points. The following are equivalent.*

1. *For any simple closed curve C, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0$.*
2. *For two simple curves C_1 and C_2 which have the same endpoints, $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$.*
3. *$\mathbf{F} = \nabla f$ for some scalar field f. If $\mathbf{F}$ has an exceptional point at which it is not defined, then f is also not defined there.*
4. *$\nabla \times \mathbf{F} = 0$.*

Proof. To show equivalence of all the conditions, a circular argument is used.

$1 \Rightarrow 2$ Consider two curves with the same endpoints, parameterised by $\boldsymbol{\oe}_1$ and $\boldsymbol{\oe}_2$. Hence $\boldsymbol{\oe} = \boldsymbol{\oe}_1 - \boldsymbol{\oe}_2$ is a closed curve, and so by (1),

$$\begin{aligned}
\int_{\boldsymbol{\oe}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= 0 \\
0 &= \int_{\boldsymbol{\oe}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{s} - \int_{\boldsymbol{\oe}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s} \\
\int_{\boldsymbol{\oe}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_{\boldsymbol{\oe}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s}
\end{aligned}$$

Hence the result.

$2 \Rightarrow 3$ Define $f(x, y, z) = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$, and now since (by hypothesis) the exact path between the origin and (x, y, z) is irrelevant, this integral can be evaluated in the following three ways. Suppose $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

- Take a path along the x axis, then parallel to the y axis, then parallel to the z axis. This integral can now be evaluated by considering the three components the path.

  For the first component, the path is given by $t\mathbf{i}$ for $0 \leqslant t \leqslant x$ which has derivative $\mathbf{i}$. The required integral is therefore

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^x \left(F_1(t,0,0)\mathbf{i} + F_2(t,0,0)\mathbf{j} + F_3(t,0,0)\mathbf{k}\right) \cdot \mathbf{i}\, \mathrm{d}t = \int_0^x F_1(t,0,0)\, \mathrm{d}t$$

  Repeating a similar process for the other two components of the path gives

$$f(x,y,z) = \int_0^x F_1(t,0,0)\, \mathrm{d}t + \int_0^y F_2(x,t,0)\, \mathrm{d}t + \int_0^z F_3(x,y,t)\, \mathrm{d}t$$

  Taking the partial derivative of this with respect to z it is evident that $\dfrac{\partial f}{\partial z} = F_3(x, y, z)$.

– Take a path along the y axis, parallel to the z axis, then parallel to the x axis. From the definition of a line integral this gives

$$f(x,y,z) = \int_0^y F_2(0,t,0)\,\mathrm{d}t + \int_0^z F_3(0,y,t)\,\mathrm{d}t + \int_0^x F_1(t,y,z)\,\mathrm{d}t$$

Taking the partial derivative of this with respect to x it is evident that $\dfrac{\partial f}{\partial x} = F_1(x,y,z)$.

– It clearly follows that $\dfrac{\partial f}{\partial y} = F_2(x,y,z)$.

Hence f has the required property for these particular paths. However, since by hypothesis any path can be chosen, the theorem holds.

$3 \Rightarrow 4$ Since $\mathbf{F}$ is the gradient of a scalar function, the result readily follows from Theorem 26

$4 \Rightarrow 1$ Let C be any closed curve, and S be a surface which has C as its boundary. (S must be chosen to avoid any exceptional points of $\mathbf{F}$.) Using Stokes' Theorem,

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{s} = \int_S \nabla\times\mathbf{F}\cdot\mathrm{d}\mathbf{S}$$

But by hypothesis $\nabla\times\mathbf{F} = 0$, and hence the result.

A circular relationship has now been established between the conditions, so they are certainly all equivalent. The truth of 1 follows from Theorem 52, when $\mathbf{F}$ is a conservative field. □

Commonly $\mathbf{F}$ is interpreted as being a force field, in which case $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{s}$ represents the "work done" in moving a particle along C. It has now been shown that this quantity only depends on the endpoints of C, and is zero if C is closed. An alternative interpretation is of $\mathbf{V}$ being a velocity, in particular of a fluid. The line integral is then called the circulation

Let $\mathbf{V}(x,y,z,t)$ be a vector field on $\mathbb{R}^3$ of class C^1, dependent upon some other parameter t. Also let $\rho(x,y,z,t)$ be a C^1 scalar field on $\mathbb{R}^3$.
Practically, $\mathbf{V}$ may be though of as a velocity, and ρ may be thought of as a density.

Definition 54 (Conservation Of Mass) *For a 'velocity' $\mathbf{V}$, 'density' ρ, and $\Omega\subset\mathbb{R}^3$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\,\mathrm{d}V = -\int_{\partial\Omega}\mathbf{J}\cdot\mathrm{d}\mathbf{S}$$

where $\mathbf{J}=\rho\mathbf{V}$ and $\partial\Omega$ is the bounding surface of Ω.

A dimensional analysis of the above equation shows that it represents a rate of mass. Notice that

- $\int_\Omega \rho\,\mathrm{d}V$ is the mass in Ω.
- $\int_{\partial\Omega}\mathbf{J}\cdot\mathrm{d}\mathbf{S}$ is the flux† of $\mathbf{J}$ and represents the rate at which mass leaves Ω.

The interpretation of the conservation law is that the rate of change of mass in Ω is equal to the rate at which mass enters Ω.

Theorem 55 *The conservation law for mass is equivalent to*

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0$$

†Physicists call a surface integral 'flux'.

Alternatively, from the definition of $\mathbf{J}$*,*

$$\rho\nabla\cdot\mathbf{V}+\mathbf{V}\cdot\nabla\cdot\rho+\frac{\partial\rho}{\partial t}=0$$

Proof. From the divergence theorem,

$$\int_{\partial\Omega}\mathbf{J}\cdot\mathrm{d}\mathbf{S}=\int_{\Omega}\nabla\cdot\mathbf{J}\,\mathrm{d}V$$

Now from the conservation of mass formula,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\,\mathrm{d}V+\int_{\partial\Omega}\mathbf{J}\cdot\mathrm{d}\mathbf{s}=0\\ \int_{\Omega}\frac{\partial\rho}{\partial t}\,\mathrm{d}V+\int_{\Omega}\nabla\cdot\mathbf{J}\,\mathrm{d}V=\int_{\Omega}\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}\,\mathrm{d}V=0\end{aligned}$$

and since this must hold for all Ω,

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$$

Hence the result. $\square$

The conservation laws often result in a particular physical situation being modeled by Laplace's equation, $\nabla^2u=0$. A number of applications of the conservation laws are now given.

(19.3.2) Laplace's Equation: The Heat Equation

Let $T(x,y,z,t)$ be the temperature at a particular time in a body, and let $\mathbf{F}=\nabla T$ be the temperature gradient. Fourier's Law states that the velocity of heat transfer is proportional to the temperature gradient. The density of thermal energy is given by $\rho=c\rho_0T$ and let $\mathbf{J}=-k\mathbf{F}$ where the constant k is called the conductivity.

If there is no heat source in Ω then the conservation law for heat must hold and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\,\mathrm{d}V=-\int_{\partial\Omega}\mathbf{J}\cdot\,\mathrm{d}S$$

and from Theorem 55

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}&=0\\ \text{so}\quad c\rho_0\frac{\partial T}{\partial t}-k\nabla^2T&=0\\ \frac{\partial T}{\partial t}&=\frac{k}{c\rho_0}\nabla^2T\end{aligned}$$

When T does not depend on time, this reduces to Laplace's equation.

(19.3.3) Laplace's Equation: Fluid Dynamics

Let $\mathbf{u}$ be the velocity of a fluid, and let ρ be the constant density of the fluid. If there are no points of mass creation or destruction (sources or sinks) then when $\mathbf{J}=\rho\mathbf{u}$,

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0$$

by conservation of mass, and now since ρ is constant it follows that $\nabla \cdot \mathbf{u} = 0$. The velocity is therefore incomprehensible—which was an assumption anyway. Assuming that $\mathbf{u}$ is also irrotational, it can be expressed as $\mathbf{u} = \nabla U$ for some scalar field U. Hence

$$\nabla^2 U = \nabla \cdot \mathbf{u} = 0$$

so Laplace's equation appears here too. It would seem that having made the assumption that $\mathbf{u}$ is both incomprehensible and irrotational would dramatically reduce the applicability of such theory. However, such 'potential flows' have many applications, including modelling water waves.

(19.3.4) Laplace's Equation: Potential Theory

Potential fields are most commonly applicable to gravitation and electrostatics. In the case of a gravitational field, recall Newton's Law Of Gravitation,

$$\mathbf{F} = \frac{-G}{r^3}\mathbf{r}$$

where $\mathbf{r}$ is the position vector between the two masses under consideration and $|\mathbf{r}| = r$. Now,

$$\nabla\left(\frac{G}{r}\right) = \nabla\left(\frac{G}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-1}{2}\frac{G}{(x^2+y^2+z^2)^{\frac{3}{2}}}(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}) = \frac{-g}{r^3}\mathbf{r} = \mathbf{F}$$

So there exists a scalar field for which $\mathbf{F}$ is the gradient, hence $\mathbf{F}$ is a potential field.

Suppose that two masses m and M are at positions (x, y, z) and (X, Y, Z) respectively, then $\mathbf{r} = (x - X)\mathbf{i} + (y - Y)\mathbf{j} + (z - Z)\mathbf{k}$. Now,

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \nabla^2\left(\frac{1}{\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}}\right) \\
&= \nabla\cdot\left(\frac{-\mathbf{r}}{r^3}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{-(x-X)}{((x-X)^2+(y-Y)^2+(z-Z)^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{-(y-Y)}{((x-X)^2+(y-Y)^2+(z-Z)^2)^{\frac{3}{2}}}\right) \\
&\qquad + \frac{\partial}{\partial z}\left(\frac{-(z-Z)}{((x-X)^2+(y-Y)^2+(z-Z)^2)^{\frac{3}{2}}}\right) \\
&= \left(\frac{3(x-X)}{r^5} - \frac{1}{r^3}\right) + \left(\frac{3(y-Y)}{r^5} - \frac{1}{r^3}\right) + \left(\frac{3(z-Z)}{r^5} - \frac{1}{r^3}\right) \\
&= \frac{3r^2}{r^5} - \frac{3}{r^3} \\
&= 0
\end{aligned}$$

From this it is evident that, since G is a constant, $\nabla^2 U = 0$ where $U = \dfrac{G}{r}$.

There is a problem when the two masses are at the same point. This can be resolved by a further analysis, but is of little concern at present. What is of interest is if the two masses are not points, but continuous distributions, like say a gas cloud may be.

Suppose that (X, Y, Z) may take values throughout some region Ω. In this case let ρ be the density of mass in Ω, giving

$$U(x, y, z) = G\iiint_\Omega \frac{\rho(X, Y, Z)}{r}\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}Z$$

Now taking the Laplacian,

$$\nabla^2 U = G \iiint_\Omega \rho(X,Y,Z) \nabla^2 \left(\frac{1}{r}\right) \, \mathrm{d}X \, \mathrm{d}Y \, \mathrm{d}Z = 0$$

Note that ∇ operates on x, y, and z, not X, Y, and Z. Hence for $(x,y,z) \notin \Omega$, $\nabla^2 U = 0$. In the case when $(x,y,z) \in \Omega$ it can be shown that $\nabla^2 U = G\rho$.

(19.4) Fourier Series & Partial Differential Equations

(19.4.1) Fourier Series

Definition 56 *For any function f of period $T = 2L$ which is defined on the interval $[-L, L]$, f has the Fourier series*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) \, \mathrm{d}x \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \end{aligned}$$

Every such function has a Fourier series, but whether or not the Fourier series converges to the function is another matter entirely.

Note that a function is piecewise continuous if it is continuous on every closed bounded interval (a, b) with the exception of finitely many points in (a, b). The integral of a piecewise continuous function over (a, b) always exists.

Theorem 57 *Let f be periodic with period 2π, be piecewise continuous on $[-\pi, \pi]$, and have a left or right (or both) derivative at each point in $[-\pi, \pi]$. If this is so, then the Fourier series converges to f wherever f is continuous, and to $\frac{f(x_0^+)+f(x_0^-}{2}$ if f is discontinuous at x_0.*

This theorem was proved in the nineteenth century by German mathematician Dirichlet. A less strict condition for convergence has not yet been found. If any of the conditions fail at some particular point then the Fourier series may not converge there, but it will still converge elsewhere. The convergence of the series therefore depends on the values x may take.

Observe that if f is even its Fourier series expansion contains only cosine terms and the coefficients are given by

$$a_0 = \frac{2}{L} \int_0^L f(x) \, \mathrm{d}x \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x$$

Similarly, if f is odd then its Fourier series expansion contains only sine terms and the coefficients are given by

$$b_n = \frac{1}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x$$

These results can be used to make Fourier series for functions which are only defined on $[0, L]$ rather than

$[-L, L]$. Extending the function into $[-L, 0]$ can be done in one of the following two ways

$$f_o(x) = \begin{cases} f(x) & 0 \leqslant x \leqslant L \\ f(-x) & -L \leqslant x \leqslant 0 \end{cases} \qquad f_e(x) = \begin{cases} f(x) & 0 \leqslant x \leqslant L \\ -f(-x) & -L \leqslant x \leqslant 0 \end{cases}$$

Clearly f_o is an odd function and so has a sine series, while f_e is an even function and so has a cosine series.

(19.4.2) Partial Differential Equations

Boundary Value Problems

In this rather simple treatment of partial differential equations—and mostly of Laplace's equation—concern lies with homogeneous linear equations. In order to find a precise solution rather than a family of solutions it is necessary to give information about a particular application of the equation to be solved.

- Conditions specifying the behavior of the solution at points in its spatial domain are called boundary conditions.
- Conditions specifying the behavior of the function at a particular time are called initial conditions.

In the simplest case, boundary conditions may be of the form $f(\mathbf{x}_0) = k$ for some constant k; these are called Dirichlet boundary conditions. Conditions of the form $\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}_0} = k$ are called Neumann boundary conditions. Obviously, calculations are simplified if $k = 0$.

A second order linear partial differential equation in two variables will most generally be of the form

$$A\frac{\partial^2 u}{\partial x_1^2} + B\frac{\partial^2 u}{\partial x_1 \partial x_2} + C\frac{\partial^2 u}{\partial x_2^2} = F\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)$$

Note that x_1 and x_2 need not be spatial variables, one could be time. Such differential equations are classified as follows.

- If $AC - B^2 > 0$ then the equation is elliptic.
- If $AC - B^2 = 0$ then the equation is parabolic.
- If $AC - B^2 < 0$ then the equation is hyperbolic.

The following three equations have very important physical interpretations, and should be noted well.

1. Laplace's equation, $\nabla^2 u = 0$ has $A = C = 1$ and $B = 0$. Clearly it is elliptic.
2. The one dimensional diffusion equation (the heat equation) is given by

   $$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

   which is inhomogeneous. $A = B = 0$ so this equation is parabolic.
3. The one dimensional wave equation is given by

   $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

   which has $A = 1$, $B = -c^2$, and $C = 0$. Clearly this is a hyperbolic equation.

An equation with variable coefficients may change between these classifications within its domain.

The Heat Equation With Dirichlet Boundary Conditions

The heat equation in one dimensions models the dispersion of heat in a bar of length L. Say the temperature at any point at any time is given by $T(x,t)$ where $0 \leqslant x \leqslant L$. The Dirichlet boundary conditions are then $T(0,t) = T_0$ and $T(L,t) = T_1$ for all t.

In general T will model temperature in a volume V bounded by a surface $S = \partial V$. The Dirichlet boundary conditions are established by saying that S is maintained at a constant temperature so that $T(S) = k$. The Neumann boundary conditions are established by saying that S is insulated so that no heat is lost. This means that where $\mathbf{n}$ is normal to S, $\nabla T \cdot \mathbf{n} = 0$ when evaluated on S, so there is no heat flux.

Worth special mention is the Steffan problem where $\nabla T \cdot \mathbf{n} = T(\mathbf{x},t)k(\mathbf{x})$ when evaluated on S and where $\mathbf{x}$ represents the spatial co-ordinates of the system.

Attention is now turned to solving the one dimensional heat equation with boundary conditions $T(0,t) = 0$ and $T(L,t) = 0$ for all t. It is also required to know that $T(x,0) = f(x)$, the initial condition. Note that in one dimension S is simply the two endpoints of the bar.

Using separation of variables, assume that the solution is of the form $T(x,t) = F(x)G(t)$. Substituting in,

$$\begin{aligned} \frac{\partial T}{\partial t} &= D\frac{\partial^2 T}{\partial x^2} \\ F(x)\frac{\mathrm{d}G}{\mathrm{d}t} &= DG\frac{\mathrm{d}^2 F}{\mathrm{d}t^2} \\ \frac{1}{DG}\frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{1}{F}\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} \end{aligned}$$

Now, the left hand side is a function of only t, and the right hand side is a function of only x. The equation can therefore only hold if the two expressions are both equal to the same constant, k, say. This gives

$$\frac{\mathrm{d}G}{\mathrm{d}t} - kDG = 0 \tag{58}$$

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - kF = 0 \tag{59}$$

First of all the x dependent equation is solved so that the first two boundary conditions can be used.

$k > 0$: Clearly the solution is

$$\begin{aligned} F(x) &= Ae^{\mu x} + Be^{-\mu x} \quad \text{where} \quad \mu = \sqrt{k} \\ \text{so } F(0) = 0 &= A + B \\ \text{and} \quad F(L) = 0 &= Ae^{\mu L} + Be^{-\mu L} \\ &= A\left(e^{\mu L} - e^{-\mu L}\right) \end{aligned}$$

Clearly $A = 0$ is possible, but is of no interest. The other possibility is that $e^{\mu L} - e^{-\mu L} = 0$. Now, $\mu^2 = k$ and $k > 0$ so in this case this equation has no solution, hence neither does equation (59).

$k = 0$: The solution is simply $F(x) = Ax + B$, and from the boundary conditions $F(0) = 0$ gives $B = 0$, and $F(L) = 0$ gives $A = 0$. Hence there is only the trivial solution.

$k < 0$: Say $k = -p^2$ so that the equation becomes

$$\begin{aligned} &\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} + p^2 F = 0 \\ &\text{giving } F(x) = A\sin(px) + B\cos(px) \end{aligned}$$

Now, $F(0) = 0$ gives $B = 0$, so to satisfy the other boundary condition, $\sin(pL) = 0$ which can happen only for specific values of p.

$$\sin(pL) = 0 \quad \text{gives} \quad p = \frac{n\pi}{L} \; (n \in \mathbb{Z}) \quad \text{so} \quad F(x) = A\sin\left(\frac{n\pi}{L}x\right) \; (n \in \mathbb{Z})$$

For the sake of argument take $A = 1$, and since this is an even function it is only really necessary to consider positive n. Hence define

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right) \text{ for } n \in \mathbb{N}$$

It follows that the constant k can be any one of $-\frac{n^2\pi^2}{L^2}$ for $n \in \mathbb{Z}$.
Returning now with knowledge about k to equation (58),

$$\frac{\mathrm{d}G_n}{\mathrm{d}t} + \lambda_n^2 G_n = 0 \quad \text{where} \quad \lambda_n^2 = Dk = D\frac{n\pi}{L}$$
$$\int \frac{1}{G_n}\,\mathrm{d}G = -\lambda_n^2 \int 1\,\mathrm{d}t$$
$$G_n = B_n e^{-\lambda_n^2 t}$$

The B_ns could be determined now, but are left for later as a useful result occurs. Since $T = FG$ it is evident that the solutions are

$$T_n(x,t) = B_n e^{-\lambda_n^2 x} \sin\left(\frac{n\pi}{L}x\right)$$

These are called the eigenfunctions with eigenvalues λ_n. Notice that these decay with time. Using the principle of supposition, in fact any linear combination of these solutions is also a solution, so

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda^2 x}$$

where the coefficients from the linear combination and the $B_n s$ have been combined into the b_ns. Now using the remaining initial condition that $T(x,0) = f(x)$,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

but clearly this is the Fourier sine series expansion of f, from which it is deduced that the values of the coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x$$

The Heat Equation With Neumann Boundary Conditions

Boundary conditions are not chosen in an arbitrary manner: they are the result of a realistic situation. The Dirichlet boundary conditions arise by maintaining the temperature along the boundary of the region under consideration to zero (or constants).

Neumann boundary conditions arise in the case when the boundary is insulated so that no heat can flow across it. For a volume V with boundary $\partial V = S$ which has normal vector $\mathbf{n}$ it must then be the case that $\nabla T \cdot \mathbf{n} = 0$.

In the case of an one dimensional bar, this gives

$$\nabla T \cdot \mathbf{n} = \frac{\partial T}{\partial x}\mathbf{i} \cdot \mathbf{i} = \frac{\partial T}{\partial x} = 0$$

which must happen at the boundary i.e. when $x = 0$ and $x = L$. As well as these conditions, it is given that $T(x, 0) = f(x)$.

To solve the heat equation $\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$ separation of variables is used again to give equations (58) and (59),

$$\frac{\mathrm{d}G}{\mathrm{d}t} - kDG = 0 \tag{58}$$
$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} - kF = 0 \tag{59}$$

For the x dependent equation, (58),

$k > 0$: The solution is of the form

$$F(x) = Ae^{\mu x} + Be^{-\mu x} \quad \text{where } \mu = \sqrt{k}$$
$$\text{giving } \frac{\mathrm{d}F}{\mathrm{d}x} = A\mu e^{\mu x} - B\mu e^{-\mu x}$$

From this it is clear that the boundary conditions give $A = B = 0$.

$k < 0$: Put $k = -p^2$ so that

$$\begin{aligned} F(x) &= A\cos(px) + B\sin(px) \\ \text{giving } \frac{\mathrm{d}F}{\mathrm{d}x} &= -Ap\sin(px) + Bp\cos(px) \\ \text{so } F'(0) &= Bp = 0 \text{ hence } B = 0 \\ \text{and } F'(L) &= -Ap\sin(pL) \\ \text{Assume } A \neq 0 \text{ giving } p &= \frac{n\pi}{L} \text{ for } n \in \mathbb{N} \\ \text{hence } F_n(x) &= B_n \cos\left(\frac{n\pi}{L}x\right) \end{aligned}$$

$k = 0$: In this case $F(x) = Ax + B$ so $F'(x) = A$ giving $A = 0$. However, B is undetermined, so calling it B_0 it can be combined with the case $k < 0$ to give the final result

$$F_n(x) = B_n \cos\left(\frac{n\pi}{L}x\right) \quad n \in \mathbb{Z}_0^+$$

Returning now to the other equation, equation (59),

$$\frac{\mathrm{d}G_n}{\mathrm{d}t} + \lambda_n^2 G_n = 0 \quad \text{where} \quad \lambda_n^2 = \sqrt{D}\frac{n\pi}{L}$$

the general solution to which is

$$G_n(t) = Ae^{-\lambda_n^2 t}$$

Hence

$$\begin{aligned} T_n(x,t) &= A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} \\ T(x,t) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} \end{aligned}$$

Now using the initial condition,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

which is in the form of a half range Fourier cosine series, hence the coefficients A_n must be given by

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x$$

Notice that at $t \to \infty$ all the terms in the sum tend to zero due to the exponential factor. This is with exception of the first term, so the temperature of the bar will approach A_0

Laplace's Equation

The Dirichlet and Neumann boundary conditions to Laplace's equation, $\nabla^2 = 0$, have the following interpretation. Consider a volume V with bounding surface $S = \partial V$.

- The Dirichlet boundary conditions specify $u(x, y, z) = f(x, y, z)$ for (x, y, z) on S. When $f = 0$ a solution can be found using Green's theorem.
- The Neumann boundary conditions specify $\frac{\mathrm{d}u}{\mathrm{d}\mathbf{n}} = \nabla u \cdot \mathbf{n} = f(x, y, z)$ for (x, y, z) on S. Some functions f will lead to the problem having no solution.
- Alternatively, a mixture of Dirichlet and Neumann conditions can be given on different parts of S.

Note that there is no time dependence in Laplace's equation. An interpretation of the boundary conditions can be made by considering a constant heat flow, with fixed temperature on S, insulation on S, and a mixture of both.

Solution To Laplace's Equation On A Disc

Consider Laplace's equation in plane polar co-ordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

and let the region of interest be a disc of radius R so tat $r = R$ is the boundary. The Neumann boundary condition is then

$$\frac{\partial u}{\partial r} = f(\theta) \quad r = R$$

Using separation of variables, assume that the solution is of the form $u(r, \theta) = F(r)G(\theta)$ hence substituting in,

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0 \\
G(\theta)F''(r) + \frac{G(\theta)F'(r)}{r} + \frac{F(r)G''(\theta)}{r^2} &= 0 \\
\frac{r^2}{F(r)}F''(r) + r\frac{F'(r)}{F(r)} &= \frac{G''(\theta)}{G(\theta)}
\end{aligned}$$

The only way this can hold is if both sides are constant, say k, giving

$$r^2F''(r) + rF'(r) = r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = kF(r) \tag{60}$$

$$G''(\theta) + kG(\theta) = 0 \tag{61}$$

Because the domain is a disc, G must be periodic with period 2π. Considering equation (61),

$k < 0$: The solution is of the form $Ae^{\theta\sqrt{k}} + Be^{-\theta\sqrt{k}}$ which is certainly not periodic and so is discounted as a possible solution.

$k > 0$: Say $p^2 = k$ then the solution is of the form

$$G(\theta) = A\cos(p\theta) + B\sin(p\theta)$$

This is periodic with period 2π whenever p is integer. Hence let

$$G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \quad n \in \mathbb{Z}_0^+$$

For equation (60) consider the substitution $s = \ln r$ giving

$$s = \ln r \qquad \frac{\mathrm{d}s}{\mathrm{d}r} = \frac{1}{r}$$

From the chain rule $\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{\mathrm{d}F}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r}$, so substituting,

$$\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F_n}{\mathrm{d}r}\right) &= n^2F_n(r) \\ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}F_n}{\mathrm{d}s}\right) &= n^2F_n(s) \\ \frac{\mathrm{d}^2F_n}{\mathrm{d}s^2} - n^2F_n(s) &= 0 \\ \text{so the solution is} \quad F_n(s) &= A_n^*e^{ns} + B_n^*e^{-ns} \quad n \neq 0 \\ \text{so } F_n(r) &= A_n^*r^n + B_n^*r^{-n} \quad n \in \mathbb{N} \end{aligned}$$

For $n = 0$, $F_0(s) = c_1s + c_2$ so $F_0(r) = c_1\ln r + c_2$. Hence for any particular n the solution is

$$u_n(r,\theta) = \begin{cases} \left(A_n^*r^n + B_n^*r^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right) & n \in \mathbb{N} \\ c_1\ln r + c_2 & n = 0 \end{cases}$$

Now using the principle of supposition,

$$u(r,\theta) = c_1\ln r + c_2 + \sum_{n=1}^{\infty}\left(A_n^*r^n + B_n^*r^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

The boundary conditions have yet to be used, and there is clearly a problem at the origin. These are now dealt with.

For problems with physical applications it is not generally possible for the solution to approach infinity at the origin. Were the domain an annulus rather than a disc this would not be a problem. To solve this problem the offending terms are removed by setting the appropriate constants to zero. Hence

$$u(r,\theta) = c_2 + \sum_{n=1}^{\infty} r^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

Note that A^* can be incorporated into A and B.

Next the boundary condition is used. In Cartesian form this was $\frac{\partial u}{\partial \mathbf{n}} = f(\theta)$. Now, clearly the normal unit vector to the boundary is $\mathbf{e}_r$ and hence the boundary condition becomes

$$\frac{\partial u}{\partial \mathbf{e}_r} = \nabla u \cdot \mathbf{e}_r = \frac{\partial u}{\partial r} = f(\theta)$$

Hence differentiating with respect to r and setting $r = R$,

$$f(\theta) = \sum_{n=1}^{\infty} nR^{n-1}\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) \tag{62}$$

Using results for the coefficients of a Fourier series it is evident that

$$\begin{aligned} A_n &= \frac{1}{nR^{n-1}} \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, \mathrm{d}\theta \\ B_n &= \frac{1}{nR^{n-1}} \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, \mathrm{d}\theta \end{aligned}$$

Observe that if the right hand side of equation (62) is integrated between 0 and 2π the result is zero. Hence a solvability criterion for the Neumann boundary conditions is that

$$\int_0^{2\pi} f(\theta)\, \mathrm{d}\theta = 0$$

This is reasonable since if Laplace's equation is interpreted as modeling heat in the disc then this criterion specifies that there is no heat flux. Also,the value of c_2 cannot be determined from the information given.